Problem 3.7
Consider a system whose Hamiltonian H and an operator A are given by the matrices

$$H = \mathcal{E}_0 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A = a \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where $\mathcal{E}_0$ has the dimensions of energy.

(a) If we measure the energy, what values will we obtain?

(b) Suppose that when we measure the energy, we obtain a value of $-\mathcal{E}_0$. Immediately afterwards, we measure A. What values will we obtain for A and what are the probabilities corresponding to each value?

(c) Calculate the uncertainty ΔA.

Solution

(a) The possible energies are given by the eigenvalues of H. A diagonalization of H yields three nondegenerate eigenenergies $E_1 = 0$, $E_2 = -\mathcal{E}_0$, and $E_3 = 2\mathcal{E}_0$. The respective eigenvectors are

$$|\phi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad |\phi_2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad |\phi_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}; \tag{3.183}$$

these eigenvectors are orthonormal.

(b) If a measurement of the energy yields $-\mathcal{E}_0$, this means that the system is left in the state $|\phi_2\rangle$. When we measure the next observable, A, the system is in the state $|\phi_2\rangle$. The result we obtain for A is given by any of the eigenvalues of A. A diagonalization of A yields three nondegenerate values: $a_1 = -\sqrt{17}a$, $a_2 = 0$, and $a_3 = \sqrt{17}a$; their respective eigenvectors are given by

$$|a_1\rangle = \frac{1}{\sqrt{34}} \begin{pmatrix} 4 \\ -\sqrt{17} \\ 1 \end{pmatrix}, \qquad |a_2\rangle = \frac{1}{\sqrt{17}} \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}, \qquad |a_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 4 \\ \sqrt{17} \\ 1 \end{pmatrix}. \tag{3.184}$$

Thus, when measuring A on a system which is in the state $|\phi_2\rangle$, the probability of finding $-\sqrt{17}a$ is given by

$$P_1(a_1) = |\langle a_1|\phi_2\rangle|^2 = \left| \frac{1}{\sqrt{34}} \begin{pmatrix} 4 & -\sqrt{17} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|^2 = \frac{1}{34}. \tag{3.185}$$

Similarly, the probabilities of measuring 0 and $\sqrt{17}a$ are

$$P_2(a_2) = |\langle a_2|\phi_2\rangle|^2 = \left| \frac{1}{\sqrt{17}} \begin{pmatrix} 1 & 0 & -4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|^2 = \frac{16}{17}, \tag{3.186}$$

$$P_3(a_3) = |\langle a_3|\phi_2\rangle|^2 = \left| \frac{1}{\sqrt{34}} \begin{pmatrix} 4 & \sqrt{17} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|^2 = \frac{1}{34}. \tag{3.187}$$

(c) Since the system, when measuring A is in the state $|\phi_2\rangle$, the uncertainty ΔA is given by $\Delta A = \sqrt{\langle\phi_2|A^2|\phi_2\rangle - \langle\phi_2|A|\phi_2\rangle^2}$, where

$$\langle\phi_2|A|\phi_2\rangle = a \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0, \tag{3.188}$$

$$\langle\phi_2|A^2|\phi_2\rangle = a^2 \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = a^2. \tag{3.189}$$

Thus we have $\Delta A = a$.

Problem 3.8

Consider a system whose state and two observables are given by

$$|\psi(t)\rangle = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \qquad A = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(a) What is the probability that a measurement of A at time t yields -1?

(b) Let us carry out a set of two measurements where B is measured first and then, immediately afterwards, A is measured. Find the probability of obtaining a value of 0 for B and a value of 1 for A.

(c) Now we measure A first then, immediately afterwards, B. Find the probability of obtaining a value of 1 for A and a value of 0 for B.

(d) Compare the results of (b) and (c). Explain.

(e) Which among the sets of operators $\{\hat{A}\}$, $\{\hat{B}\}$, and $\{\hat{A}, \hat{B}\}$ form a complete set of commuting operators (CSCO)?

Solution

(a) A measurement of A yields any of the eigenvalues of A which are given by $a_1 = -1$, $a_2 = 0$, $a_3 = 1$; the respective (normalized) eigenstates are

$$|a_1\rangle = \frac{1}{2} \begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix}, \qquad |a_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \qquad |a_3\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}. \tag{3.190}$$

The probability of obtaining $a_1 = -1$ is

$$P(-1) = \frac{|\langle a_1|\psi(t)\rangle|^2}{\langle\psi(t)|\psi(t)\rangle} = \frac{1}{6} \left| \frac{1}{2} \begin{pmatrix} -1 & \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right|^2 = \frac{1}{3}, \tag{3.191}$$

where we have used the fact that $\langle\psi(t)|\psi(t)\rangle = \begin{pmatrix} -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 6$.

(b) A measurement of B yields a value which is equal to any of the eigenvalues of B: $b_1 = -1$, $b_2 = 0$, and $b_3 = 1$; their corresponding eigenvectors are

$$|b_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad |b_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |b_3\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{3.192}$$

Since the system was in the state $|\psi(t)\rangle$, the probability of obtaining the value $b_2 = 0$ for B is

$$P(b_2) = \frac{|\langle b_2|\psi(t)\rangle|^2}{\langle\psi(t)|\psi(t)\rangle} = \frac{1}{6}\left|\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}\right|^2 = \frac{2}{3}. \tag{3.193}$$

We deal now with the measurement of the other observable, A. The observables A and B do not have common eigenstates, since they do not commute. After measuring B (the result is $b_2 = 0$), the system is left, according to Postulate 3, in a state $|\phi\rangle$ which can be found by projecting $|\psi(t)\rangle$ onto $|b_2\rangle$:

$$|\phi\rangle = |b_2\rangle\langle b_2|\psi(t)\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}. \tag{3.194}$$

The probability of finding 1 when we measure A is given by

$$P(a_3) = \frac{|\langle a_3|\phi\rangle|^2}{\langle\phi|\phi\rangle} = \frac{1}{4}\left|\frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}\right|^2 = \frac{1}{2}, \tag{3.195}$$

since $\langle\phi|\phi\rangle = 4$. In summary, when measuring B then A, the probability of finding a value of 0 for B and 1 for A is given by the product of the probabilities (3.193) and (3.195):

$$P(b_2, a_3) = P(b_2)P(a_3) = \frac{2}{3}\frac{1}{2} = \frac{1}{3}. \tag{3.196}$$

(c) Next we measure A first then B. Since the system is in the state $|\psi(t)\rangle$, the probability of measuring $a_3 = 1$ for A is given by

$$P'(a_3) = \frac{|\langle a_3|\psi(t)\rangle|^2}{\langle\psi(t)|\psi(t)\rangle} = \frac{1}{6}\left|\frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}\right|^2 = \frac{1}{3}, \tag{3.197}$$

where we have used the expression (3.190) for $|a_3\rangle$.

We then proceed to the measurement of B. The state of the system just after measuring A (with a value $a_3 = 1$) is given by a projection of $|\psi(t)\rangle$ onto $|a_3\rangle$:

$$|\chi\rangle = |a_3\rangle\langle a_3|\psi(t)\rangle = \frac{1}{4}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}\begin{pmatrix} 1 & \sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}. \tag{3.198}$$

So the probability of finding a value of $b_2 = 0$ when measuring B is given by

$$P'(b_2) = \frac{|\langle b_2|\chi\rangle|^2}{\langle\chi|\chi\rangle} = \frac{1}{2}\left|\frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}\right|^2 = \frac{1}{2}, \tag{3.199}$$

since $\langle\chi|\chi\rangle = 2$.

So when measuring A then B, the probability of finding a value of 1 for A and 0 for B is given by the product of the probabilities (3.199) and (3.197):

$$P(a_3, b_2) = P'(a_3)P'(b_2) = \frac{1}{3}\frac{1}{2} = \frac{1}{6}. \tag{3.200}$$

(d) The probabilities $P(b_2, a_3)$ and $P(a_3, b_2)$, as shown in (3.196) and (3.200), are different. This is expected, since A and B do not commute. The result of the successive measurements of A and B therefore depends on the order in which they are carried out. The probability of obtaining 0 for B then 1 for A is equal to $\frac{1}{3}$. On the other hand, the probability of obtaining 1 for A then 0 for B is equal to $\frac{1}{6}$. However, if the observables A and B commute, the result of the measurements will not depend on the order in which they are carried out (this idea is illustrated in the following solved problem).

(e) As stated in the text, any operator with non-degenerate eigenvalues constitutes, all by itself, a CSCO. Hence each of $\{\hat{A}\}$ and $\{\hat{B}\}$ forms a CSCO, since their eigenvalues are not degenerate. However, the set $\{\hat{A}, \hat{B}\}$ does not form a CSCO since the opertators $\{\hat{A}\}$ and $\{\hat{B}\}$ do not commute.

Problem 3.9

Consider a system whose state and two observables A and B are given by

$$|\psi(t)\rangle = \frac{1}{6}\begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \qquad A = \frac{1}{\sqrt{2}}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$$

(a) We perform a measurement where A is measured first and then, immediately afterwards, B is measured. Find the probability of obtaining a value of 0 for A and a value of 1 for B.

(b) Now we measure B first then, immediately afterwards, A. Find the probability of obtaining a value of 1 for B and a value of 0 for A.

(c) Compare the results of (b) and (c). Explain.

(d) Which among the sets of operators $\{\hat{A}\}$, $\{\hat{B}\}$, and $\{\hat{A}, \hat{B}\}$ form a complete set of commuting operators (CSCO)?

Solution

(a) A measurement of A yields any of the eigenvalues of A which are given by $a_1 = 0$ (not degenerate) and $a_2 = a_3 = 2$ (doubly degenerate); the respective (normalized) eigenstates are

$$|a_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}, \qquad |a_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}, \qquad |a_3\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{3.201}$$

The probability that a measurement of A yields $a_1 = 0$ is given by

$$P(a_1) = \frac{|\langle a_1|\psi(t)\rangle|^2}{\langle\psi(t)|\psi(t)\rangle} = \frac{36}{17}\left|\frac{1}{\sqrt{2}}\frac{1}{6}\begin{pmatrix} 0 & i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}\right|^2 = \frac{8}{17}, \tag{3.202}$$

where we have used the fact that $\langle\psi(t)|\psi(t)\rangle = \frac{1}{36}\begin{pmatrix} 1 & 0 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} = \frac{17}{36}$.