Zeros and ties [edit]

In real data, it sometimes happens that there is a sample X_i which equals zero or a pair (X_i, Y_i) with $X_i = Y_i$. It can also happen that there are tied samples. This means that for some $i \neq j$, we have $X_i = X_j$ (in the one-sample case) or $X_i - Y_i = X_j - Y_j$ (in the paired sample case). This is particularly common for discrete data. When this happens, the test procedure defined above is usually undefined because there is no way to uniquely rank the data. (The sole exception is if there is a single sample X_i which is zero and no other zeros or ties.) Because of this, the test statistic needs to be modified.

Zeros [edit]

Wilcoxon's original paper did not address the question of observations (or, in the paired sample case, differences) that equal zero. However, in later surveys, he recommended removing zeros from the sample.[29] Then the standard signed-rank test could be applied to the resulting data, as long as there were no ties. This is now called the *reduced sample procedure*.

Pratt[30] observed that the reduced sample procedure can lead to paradoxical behavior. He gives the following example. Suppose that we are in the one-sample situation and have the following thirteen observations:

0, 2, 3, 4, 6, 7, 8, 9, 11, 14, 15, 17, −18.

The reduced sample procedure removes the zero. To the remaining data, it assigns the signed ranks:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, −12.

This has a one-sided *p*-value of $55/2^{12}$, and therefore the sample is not significantly positive at any significance level $\alpha < 55/2^{12} \approx 0.0134$. Pratt argues that one would expect that decreasing the observations should certainly not make the data appear more positive. However, if the zero observation is decreased by an amount less than 2, or if all observations are decreased by an amount less than 1, then the signed ranks become:

−1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, −13.

This has a one-sided *p*-value of $109/2^{13}$. Therefore the sample would be judged significantly positive at any significance level $\alpha > 109/2^{13} \approx 0.0133$. The paradox is that, if α is between $109/2^{13}$ and $55/2^{12}$, then *decreasing* an insignificant sample causes it to appear significantly *positive*.

Pratt therefore proposed the *signed-rank zero procedure*. This procedure includes the zeros when ranking the samples. However, it excludes them from the test statistic, or equivalently it defines $\operatorname{sgn}(0) = 0$. Pratt proved that the signed-rank zero procedure has several desirable behaviors not shared by the reduced sample procedure:[31]

1. Increasing the observed values does not make a significantly positive sample insignificant, and it does not make an insignificant sample significantly negative.
2. If the distribution of the observations is symmetric, then the values of μ which the test does not reject form an interval.
3. A sample is significantly positive, not significant, or significantly negative, if and only if it is so when the zeros are assigned arbitrary non-zero signs, if and only if it is so when the zeros are replaced with non-zero values which are smaller in absolute value than any non-zero observation.
4. For a fixed significance threshold α, and for a test which is randomized to have level exactly α, the probability of calling a set of observations significantly positive (respectively, significantly negative) is a non-decreasing (respectively, non-increasing) function of the observations.

Pratt remarks that, when the signed-rank zero procedure is combined with the average rank procedure for resolving ties, the resulting test is a consistent test against the alternative hypothesis that, for all $i \neq j$, $\Pr(X_i + X_j > 0)$ and $\Pr(X_i + X_j < 0)$ differ by at least a fixed constant that is independent of i and j.[32]