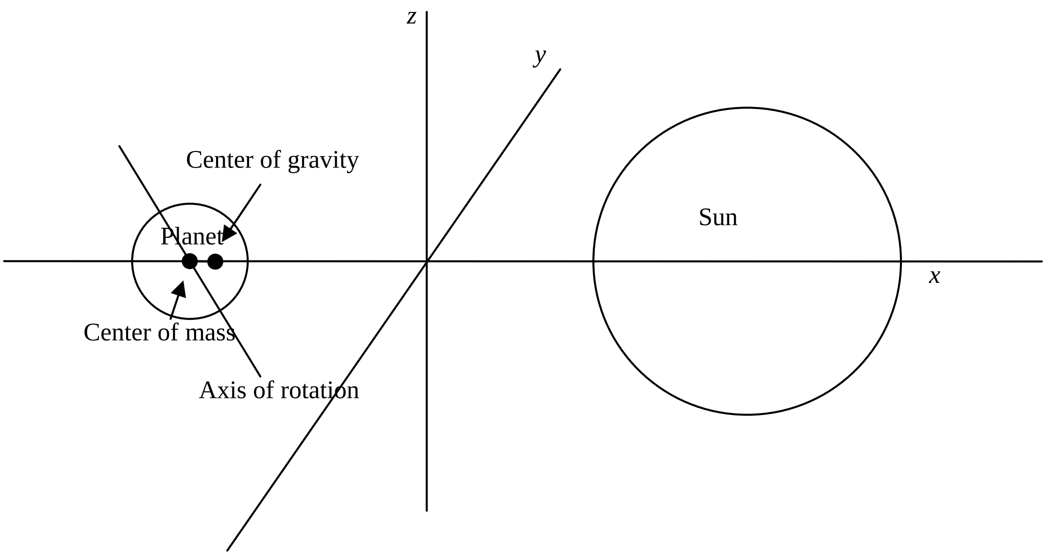
Torque on a spherical planet caused by the influence of another mass:

Theory:

The center of gravity in a spherical planet rotating around a sun does not coincide with the center of mass of the planet—due to the non-uniform nature of Newton's law of gravitation. Therefore, a torque is produced by the sun when the planet spins around a tilted axial.

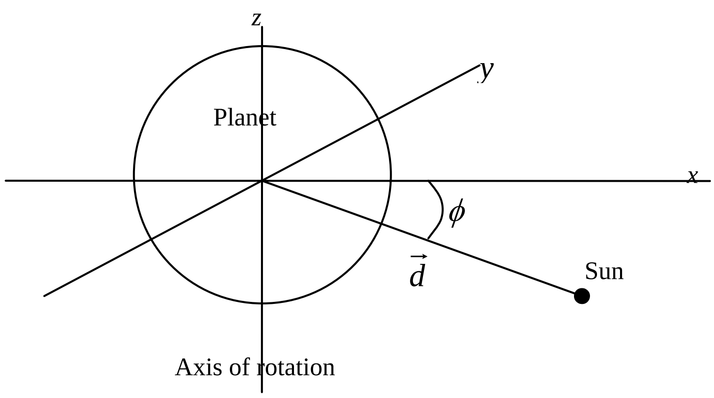
Reasons:

Due to the non-uniformity of Newton's law of gravitation, different parts of the planets do not experience the same forces. The parts that are further away from the sun experience less forces, hence the torques do not cancel out perfectly. Thus when a planet spins around a axis that is neither parallel nor perpendicular to the position vector between the center of mass of the two bodies, a non-zero toque is resulted.



Because of symmetry, the center of gravity and the center of mass can only differ in the *x* direction. Hence the torque produced by the Sun must be in the *y* direction (otherwise a non-spinning planet would begin to spin and keep gaining rotational energy indefinitely)

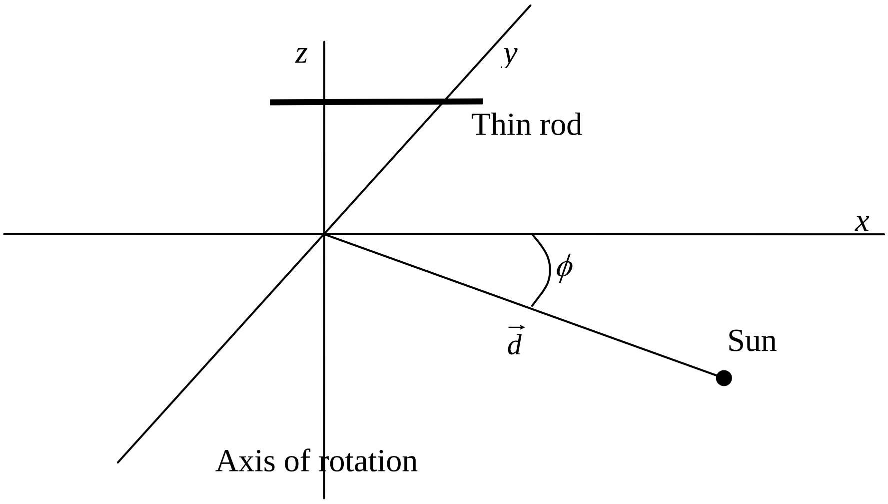
We can rotate the *xz*-plane and consider the sun as a point mass concentrated at its center of mass to simplify our calculations:



The total torque on the sphere is:

$$\vec{\tau}_{net} = GM \iiint_{sphere} \langle x, y, 0\rangle \times \frac{\langle d\cos\phi, 0, -d\sin\phi\rangle - \langle x, y, z\rangle}{\left\|\langle d\cos\phi, 0, -d\sin\phi\rangle - \langle x, y, z\rangle\right\|^3} dm$$

Instead of directly solving this integral, let us integrate through a thin rod, a disk then the whole sphere.



The integral would be:

$$\vec{\tau}_{net} = \int_{x=-X}^{x=X} \vec{r}_{moment} \times \vec{g} dm \tag{1}$$

Assume that the rod is uniform with linear density λ; the differential mass is:

$$dm = \lambda dx \tag{2}$$

Let $\vec{r}$ be the position vector of each differential mass.
Thus, according to Newton's law of gravitation, we have:

$$\vec{g} = \frac{GM\left(\vec{d} - \vec{r}\right)}{\left\|\vec{d} - \vec{r}\right\|^3}$$

$$\vec{d} - \vec{r} = \langle d\cos\phi, 0, -d\sin\phi\rangle - \langle x, y, z\rangle = \langle d\cos\phi - x, -y, -d\sin\phi - z\rangle \tag{3}$$

$$\left\|\vec{d} - \vec{r}\right\| = \sqrt{(d\cos\phi - x)^2 + (-y)^2 + (-d\sin\phi - z)^2} = \sqrt{z^2 + 2dz\sin\phi - 2dx\cos\phi + d^2 + x^2 + y^2}$$

Since the axis of spinning is the z-axis, $\vec{r}_{moment}$ consists of only the x and the y components of the position vector, hence:

$$\vec{r}_{torque} = \langle x, y, 0\rangle \tag{4}$$

Substitute (2), (3) and (4) into (1):

$$\vec{\tau}_{rod,net} = GM\lambda \int_{x=-X}^{x=X} \langle x, y, 0\rangle \times \frac{\langle d\cos\phi - x, -y, -d\sin\phi - z\rangle}{\sqrt{z^2 + 2dz\sin\phi - 2dx\cos\phi + d^2 + x^2 + y^2}^{3}} dx \qquad (5)$$

Since the torque on the sphere must be in the *y* direction, only the *y* component of the integral is needed.

We use determinate to find out the *y* component of $\langle x, y, 0\rangle \times \langle d\cos\phi - x, -y, -d\sin\phi - z\rangle$:

$$\left|\begin{matrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & 0 \\ d\cos\phi - x & -y & -d\sin\phi - z \end{matrix}\right| \begin{matrix} \hat{x} & \hat{y} \\ x & 0 \\ d\cos\phi - x & -y \end{matrix} \qquad (6)$$

$$\vec{y}_{cross} = x(d\sin\phi + z) = d\sin\phi\, x + xz$$

Substitute the result in (6) into (5):

$$\tau_{rod,z} = GM\lambda \int_{x=-X}^{x=X} \frac{(d\sin\phi\, x + xz)dx}{\left(z^2 + 2dz\sin\phi - 2dx\cos\phi + d^2 + x^2 + y^2\right)^{\frac{3}{2}}}$$

$$\tau_{rod,z} = GM\lambda(d\sin\phi + z) \int_{x=-X}^{x=X} \frac{xdx}{\left(z^2 + 2dz\sin\phi - 2dx\cos\phi + d^2 + x^2 + y^2\right)^{\frac{3}{2}}} \qquad (7)$$

The integral is in the form of:

$$\int_{x=-X}^{x=X} \frac{xdx}{\left(x^2 - Bx + C\right)^{\frac{3}{2}}}$$

$$B = 2d\cos\phi$$

$$C = 2dy\sin\phi + d^2 + y^2 + z^2$$

Which can be solved by completing the square and u-substitutions:

$$\int_{x=-X}^{x=X} \frac{xdx}{\left(x^2 - Bx + C\right)^{\frac{3}{2}}} = \int_{x=-X}^{x=X} \frac{xdx}{\left[\left(x - \frac{B}{2}\right)^2 - \frac{B^2}{4} + C\right]^{\frac{3}{2}}}$$

*notice that this integral is non-zero, since the denominator results in a graph that is greater when *x* is positive. If $d\sin\phi$ is significant enough so that $d\sin\phi + z > 0$ is always true, then every differential rod will have a positive “contribution” to the torque when integrated again and again. Hence the total net torque on the sphere must be non-zero.

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