Ternary Universal ZF Set Theory

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this is a rough draft of an outline to remove russell's paradox which prohibits the existence of a so-called universal set, a set of all sets, by using three valued, or ternary, logic. we could use fuzzy logic but ternary logic is sufficient for this task and we will not treat the general case here.

0.0.1 ternary logic

let f be a function whose domain is all well formed formulas (wffs) and whose range is $\{F, M, T\}$ that satisfies the following properties for all wffs A and B:

$f(A \vee B) = f(B \vee A) = \max\{f(A), f(B)\}$, where the order used is $F < M < T$,

$f(\neg A) = T$ if $f(A) = F$, $f(\neg A) = M$ if $f(A) = M$, and $f(\neg A) = F$ if $f(A) = T$. more generally, one can define $f(A \vee B)$ however one wishes as long as it gives T if at least one of $f(A)$ and $f(B)$ are T and F if they are both F.

these rules lead to rules for $\wedge$, $\rightarrow$, and $\leftrightarrow$ in the following way: $A \wedge B := \neg(\neg A \vee \neg B)$, $A \rightarrow B := \neg A \vee B$, and $A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A)$.

definition a perspective, or a ternary extension of binary logic, is any function f that satisfies the above relations for $\vee$ and $\neg$ and agrees with binary logic when the range is restricted to $\{F, T\}$.

here we shall use the perspective V such that if $V(A) = M$, then $V(\neg A) = M$ and $V(A \vee B) = \max\{V(A), V(B)\}$. it turns out that $V(A \wedge B) = \min\{V(A), V(B)\}$ and $V(\neg A) = 1 - V(A)$ if one treats F like 0, M like 1/2, and T like 1. this generalizes to fuzzy logic.

for reference, i will include some truth tables in ternary logic:

A	B	$\neg A$	$A \vee B$	$A \wedge B$	$A \to B$	$A \leftrightarrow B$	$(A \wedge (A \to B)) \to B$
T	T	F	T	T	T	T	T
T	M	F	T	M	M	M	M
T	F	F	T	F	F	F	T
M	T	M	T	M	T	M	T
M	M	M	M	M	M	M	M
M	F	M	M	F	M	M	M
F	T	T	T	F	T	F	T
F	M	T	M	F	T	M	T
F	F	T	F	F	T	T	T

note that the standard modus ponens deductive rule is *not* a tautology but that will not prove to be a serious problem when we apply ternary logic to set theory because we will minimize the fuzziness of our approach.

we will now introduce two new logical connectives $\to_+$ and $\leftrightarrow_+$, which generalize the binary connectives they resemble, by a truth table:

A	B	$A \to_+ B$	$A \leftrightarrow_+ B$	$A \leftrightarrow_+ \neg A$
T	T	T	T	F
T	M	T	T	F
T	F	F	F	F
M	T	T	T	T
M	M	T	T	T
M	F	T	T	T
F	T	T	F	F
F	M	T	T	F
F	F	T	T	F

0.0.2 ternary zfc (zermelo-fraenkel-choice set theory)

we will define four membership symbols to add to the language of set theory in the context of ternary logic. in what follows, all constants and variables are sets.

definition $x \in_? y$ means that $V(x \in y)$, where this $\in$ is the usual set theory symbol, is unknown or unspecified.

definition $x \in y$, a new $\in$ symbol, means that $V(x \in y)$, where this $\in$ is the usual set theory symbol, is equal to T. note that by using the new symbol, $V(x \in y) \in \{F, T\}$; i.e., it is a crisp object.

definition $x \in_M y$ means that $V\,(x \in y)$, where this $\in$ is the usual set theory symbol, is equal to M.

definition $x \notin y$ means that $V\,(x \in y)$, where this $\in$ is the usual set theory symbol, is equal to F.

definition y is fuzzy if there is a x such that $x \in_M y$. otherwise, it is crisp.

definition $x \subset y$ means that $\forall z\,(z \in x \rightarrow z \in y)$. note that with the new $\in$ notation, this is a crisp wff.

we will keep most of the axioms as they are with two exceptions which is an extension of the subsets axiom called SA2, where A is a wff:

SA2 $\forall a \exists x \forall y\,(y \in_? x \leftrightarrow_+ y \in_? a \wedge A\,(y))$,

and the axiom of foundation extension called F2:

F2 $\forall a\,((a \neq \emptyset \wedge (\exists U \forall y\,(y \in U) \rightarrow U \notin a)) \rightarrow \exists x \in a\,(x \cap a = \emptyset))$.

without saying anything further, F2 is the same as the axiom of foundation because $\neg \exists U \forall y\,(y \in U)$.

some detailed remarks on S2 are in order. the subsets axiom is $\forall a \exists x \forall y\,(y \in x \leftrightarrow y \in a \wedge A\,(y$ we will give a truth table for the wffs $y \in x$, $y \in a \wedge A\,(y)$, and $y \in x \leftrightarrow y \in a \wedge A\,(y)$ followed by the truth table for $y \in_? x$, $y \in_? a \wedge A\,(y)$, and $y \in_? x \leftrightarrow_+ y \in_? a \wedge A\,(y)$ were those all crisp objects:

$y \in x$	$y \in a \wedge A\,(y)$	$y \in x \leftrightarrow y \in a \wedge A\,(y)$
T	T	T
T	F	F
F	T	F
F	F	T

$y \in_? x$	$y \in_? a \wedge A\,(y)$	$y \in_? x \leftrightarrow_+ y \in_? a \wedge A\,(y)$
T	T	T
T	F	F
F	T	F
F	F	T

.

as you can see, they are identical which means that S2 is a generalization of the subsets axiom and *set theory will be unchanged when applied to crisp* *object in S2*. one is free to generalize it in any way one wishes but such generalizations are either useful or useless. this generalization removes russell's paradox. first, we will give the full truth table to update the last one:

$y \in_? x$	$y \in_? a \wedge A(y)$	$y \in_? x \leftrightarrow_+ y \in_? a \wedge A(y)$
T	T	T
T	M	T
T	F	F
M	T	T
M	M	T
M	F	T
F	T	F
F	M	T
F	F	T

.

both a drawback and a boon is that if $V(y \in_? a \wedge A(y)) = M$, this gives no information on $V(y \in_? x)$: you can see that in the three cases when $V(y \in_? a \wedge A(y)) = M$, $V(y \in_? x)$ can be F, M, or T with the plus-biconditional being T.

0.0.3 tuzfc (ternary universal zermelo-fraenkel-choice set theory)

so far we have done nothing of particular interest. now we introduce the universal set axiom:

universal set axiom $\exists x \forall y (y \in x)$.

we will denote this x, and it is unique, by U. we will now show how russell's paradox fails to be a paradox. let $S_1 = \{x \in U : x \notin x\}$ and $S_2 = \{x \in U : \neg(x \in_? x)\}$. then $S_1 \in_? S_1 \leftrightarrow_+ S_1 \notin S_1$. the table below shows that $S_1 \in_M S_1$.

$S_1 \in_? S_1$	$S_1 \notin S_1$	$S_1 \in_? S_1 \leftrightarrow_+ S_1 \notin S_1$
T	F	F
M	F	T
F	T	F

similarly, $S_2 \in_? S_2 \leftrightarrow_+ \neg(S_2 \in_? S_2)$ and the table below shows that $S_2 \in_M S_2$.

$S_2 \in_? S_2$	$\neg(S_2 \in_? S_2)$	$S_2 \in_? S_2 \leftrightarrow_+ \neg(S_2 \in_? S_2)$
T	F	F
M	M	T
F	T	F

.

thus, russell's paradox is no longer a paradox. however, this provides an existence proof of there being any fuzzy sets, namely S_1 and S_2.

the foundation axiom 2 now applies and we can restate it this way:

F2 $\forall a\,((a \neq \emptyset \wedge U \notin a) \rightarrow \exists x \in a\,(x \cap a = \emptyset))$.