Problem. A solid hemisphere is placed on a horizontal ledge so that its center is directly above the edge, fixed there by a delicate thread. It is given a push of negligible velocity and begins to topple over the edge. It undergoes a perfectly inelastic collision with the vertical face of the ledge and the thread breaks. What is the translational kinetic energy at this moment?

Solution. Initially the hemisphere has only gravitational potential energy measured relative to the ledge (its kinetic energy is negligible), which is converted to kinetic energy at the moment of collision. The gravitational potential energy is given by $U_g = Mgr_{\mathrm{CM}}$ where r_{CM} is the height of the center of mass above the geometric center. We now need to find the center of mass of a solid hemisphere.

Clearly the CM lies on the axis of symmetry, at a height given by $r_{\mathrm{CM}} = \frac{1}{M}\int z\,\mathrm{d}m$ where z is the height of the mass $\mathrm{d}m$ above the geometric center. The masses are circular cross sections of radius $r = \sqrt{R^2 - z^2}$ and thickness $\mathrm{d}z$, and the mass density is $\rho = 3M/(2\pi R^3)$, so

$$r_{\mathrm{CM}} = \frac{1}{M}\int_0^R z\rho\pi(R^2 - z^2)\,\mathrm{d}z = \frac{3}{2R^3}[\tfrac{1}{2}R^2z^2 - \tfrac{1}{4}z^4]_0^R = \tfrac{3}{8}R.$$

The rotational moment of inertia of the hemisphere about its geometric center is $\frac{2}{5}MR^2$, so its kinetic energy immediately before the collision is $K = \frac{1}{5}MR^2\omega_0^2$. Thus, energy conservation gives

$$\tfrac{3}{8}MgR = \tfrac{1}{5}MR^2\omega_0^2 \Rightarrow \omega_0 = \sqrt{\frac{15g}{8R}}.$$

After the collision, the hemisphere is in free-fall and so rotates about its center of mass. We therefore need the moment of inertia I_0 of a hemisphere about its center of mass. By the parallel-axis theorem,

$$I_{\mathrm{center}} = I_0 + Md^2 \Rightarrow \tfrac{2}{5}MR^2 = I_0 + M(\tfrac{3}{8}R)^2 \Rightarrow I_0 = \tfrac{2}{5}MR^2 - \tfrac{9}{64}MR^2 = \tfrac{83}{320}MR^2.$$

Let Δp be the impulse delivered to the hemisphere by the ledge. Due to perfect inelasticity, we can assume the impulse is delivered to the lowest point of the hemisphere, hence at perpendicular distance R relative to the center of mass. The final horizontal linear and angular velocities are given by

$$Mv = \Delta p, \qquad \tfrac{83}{32}MR^2\omega = \tfrac{1}{5}MR^2\omega_0 - R\Delta p.$$

The speed of the lowest point of the hemisphere relative to the CM is $\frac{\sqrt{73}}{8}R\omega$, so the horizontal component is $v = R\omega$. Therefore,

$$\tfrac{83}{320}MR^2\omega = \tfrac{1}{5}MR^2\omega_0 - R\Delta p = \tfrac{1}{5}MR^2\omega_0 - R(Mv) = \tfrac{1}{5}MR^2\omega_0 - MR\omega \Rightarrow \tfrac{403}{320}\omega = \tfrac{1}{5}\omega_0 \Rightarrow \omega = \tfrac{64}{403}\omega_0.$$

There is horizontal linear velocity $v_x = \frac{64}{403}R\omega_0$ and vertical linear velocity $v_y = -\frac{3}{8}R\omega_0$; hence the translational kinetic energy is

$$K^{(\mathrm{trans})} = \tfrac{1}{2}M(v_x^2 + v_y^2) = \tfrac{1}{2}M\big([\tfrac{64}{403}]^2 + [\tfrac{3}{8}]^2\big)R^2\omega_0^2 = \tfrac{1}{2}\big(\tfrac{4{,}096}{162{,}409} + \tfrac{9}{64}\big)M(\tfrac{15}{8}gR)$$

$$= \frac{(4,096 \cdot 64 + 9 \cdot 162,409) \cdot 15}{1,024 \cdot 162,409}MgR = \boxed{\frac{25,857,375}{166,306,816}MgR.}$$
