and the real roots are presented in the form

$$y_1 = \sqrt[3]{\omega} + \sqrt[3]{\omega^2},$$
$$y_2 = \omega\sqrt[3]{\omega} + \omega^2\sqrt[3]{\omega^2},$$
$$y_3 = \omega^2\sqrt[3]{\omega} + \omega\sqrt[3]{\omega^2}.$$

These expressions are not suitable for direct calculation because of the cube roots of imaginary numbers. If we try to find

$$\sqrt[3]{\omega} = a + bi$$

algebraically, we are led to the solution of the two simultaneous equations

$$a^3 - 3ab^2 = -\frac{1}{2}, \qquad 3a^2b - b^3 = \frac{\sqrt{3}}{2}.$$

Solving for b^2 in the first and substituting

$$b^2 = \frac{2a^3 + 1}{6a}$$

into the second, we find

$$b\left(3a^2 - \frac{2a^3 + 1}{6a}\right) = \frac{\sqrt{3}}{2},$$

whence

$$b = \frac{3\sqrt{3}a}{16a^3 - 1}, \qquad b^2 = \frac{27a^2}{(16a^3 - 1)^2}.$$

Equating the two expressions for b^2, we have the equation

$$\frac{2a^3 + 1}{6a} = \frac{27a^2}{(16a^3 - 1)^2},$$

which after due simplifications becomes

$$(2a)^9 + 3(2a)^6 - 24(2a)^3 + 1 = 0.$$

Setting $x = 8a^3$, we have for x a cubic equation

$$x^3 + 3x^2 - 24x + 1 = 0.$$

which by the substitution $x = y - 1$ is transformed into

$$y^3 - 27y - 27 = 0;$$

or, setting $y = -3z$, into

$$z^3 - 3z + 1 = 0.$$

But this is the same equation that we wanted to solve. Consequently, we did not advance a step in trying to find a and b by an algebraic process. The fact that the real roots of a cubic equation

$$y^3 + py + q = 0$$

in case

$$4p^3 + 27q^2 < 0$$

are presented in a form involving the cube roots of imaginary numbers puzzled the old algebraists for a long time, and this case was called by them *casus irreducibilis*, irreducible case. We know now that, for instance, when p and q are rational numbers, but among the three real roots of an equation

$$y^3 + py + q = 0$$