4. Irreducible Case. We return now to the discussion of the general solution and consider what happens when $\Delta < 0$. A curious phenomenon occurs here, for in this case

$$\sqrt{\frac{q^2}{4}+\frac{p^3}{27}} = i\sqrt{\frac{-\Delta}{108}}$$

is purely imaginary and both numbers

$$A = -\frac{q}{2} + i\sqrt{\frac{-\Delta}{108}}, \qquad B = -\frac{q}{2} - i\sqrt{\frac{-\Delta}{108}}$$

are imaginary, so that the roots of equation (1) in Sec. 2 are expressed through the cube roots of imaginary numbers, and yet all three of them are real. To see this let

$$\sqrt[3]{A} = a + bi$$

be one of the cube roots of A. Since B is conjugate to A, the number $a - bi$ will be one of the cube roots of B, and it must be taken equal to $\sqrt[3]{B}$ in order to satisfy the condition

$$\sqrt[3]{A}\cdot\sqrt[3]{B} = -\frac{p}{3}.$$

Thus,

$$\sqrt[3]{A} = a + bi, \qquad \sqrt[3]{B} = a - bi,$$

and from Cardan's formulas it follows that the roots

$$\begin{aligned} y_1 &= 2a,\\ y_2 &= (a+bi)\omega + (a-bi)\omega^2 = -a - b\sqrt{3},\\ y_3 &= (a+bi)\omega^2 + (a-bi)\omega = -a + b\sqrt{3} \end{aligned}$$

are real and, moreover, unequal. It is clear that $y_2 \neq y_3$. If $y_1 = y_2$, we should have

$$b = -a\sqrt{3},$$

so that

$$\sqrt[3]{A} = a(1 - i\sqrt{3}).$$

But then

$$A = a^3(1 - i\sqrt{3})^3 = -8a^3$$

would be real, which is not true. Similarly, it is shown that $y_1 \neq y_3$.

Example. To solve the equation

$$y^3 - 3y + 1 = 0.$$

In this case

$$p = -3, \qquad q = 1, \qquad \Delta = -81, \qquad \sqrt{\frac{-\Delta}{108}} = \frac{\sqrt{3}}{2},$$

$$A = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = \omega, \qquad B = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = \omega^2.$$