Exercise 5.4. INERTIAL MASS PER UNIT VOLUME

Consider a stressed medium in motion with ordinary velocity $|v| \ll 1$ with respect to a specific Lorentz frame.

(a) Show by Lorentz transformations that the spatial components of the momentum density are

$$T^{0j} = \sum_k m^{jk} v^k, \tag{5.51}$$

where

$$m^{jk} = T^{\bar{0}\bar{0}}\,\delta^{jk} + T^{\bar{j}\bar{k}} \tag{5.52}$$

and $T^{\bar{\mu}\bar{\nu}}$ are the components of the stress-energy tensor in the rest frame of the medium. Throughout the solar system $T^{\bar{0}\bar{0}} \gg |T^{\bar{j}\bar{k}}|$ (see, e.g., discussion of hurricane in §5.10), so one is accustomed to write $T^{0j} = T^{\bar{0}\bar{0}} v^j$, i.e., "(momentum density) = (rest-mass density) $\times$ (velocity)". But inside a neutron star $T^{\bar{0}\bar{0}}$ may be of the same order of magnitude as $T^{\bar{j}\bar{k}}$, so one must replace "(momentum density) = (rest-mass density) $\times$ (velocity)" by equations (5.51) and (5.52), at low velocities.

(b) Derive equations (5.51) and (5.52) from Newtonian considerations plus the equivalence of mass and energy. (*Hint:* the total mass-energy carried past the observer by a volume V of the medium includes both the rest mass $T^{\bar{0}\bar{0}}V$ and the work done by forces acting across the volume's faces as they "push" the volume through a distance.)

(c) As a result of relation (5.51), the force per unit volume required to produce an acceleration dv^k/dt in a stressed medium, which is at rest with respect to the man who applies the force, is

$$F^j = dT^{0j}/dt = \sum_k m^{jk}\, dv^k/dt. \tag{5.53}$$

This equation suggests that one call m^{jk} the "inertial mass per unit volume" of a stressed medium at rest. In general m^{jk} is a symmetric 3-tensor. What does it become for the special case of a perfect fluid?

(d) Consider an isolated, stressed body at rest and in equilibrium ($T^{\alpha\beta}{}_{,0} = 0$) in the laboratory frame. Show that its total inertial mass, defined by

$$M^{ij} = \int_{\substack{\text{stressed}\\ \text{body}}} m^{ij}\, dx\, dy\, dz, \tag{5.54}$$

is isotropic and equals the rest mass of the body

$$M^{ij} = \delta^{ij} \int T^{00}\, dx\, dy\, dz. \tag{5.55}$$