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1 The interaction picture

s.intpic

In the Schrödinger picture, a wave function $\psi = \psi(t)$ (time-dependence of wave functions and operators is suppressed, except in initial conditions) evolves according to the Schrödinger equation

$$i\hbar\dot{\psi} = H\psi,$$

where H is the Hamiltonian of the system of interest. To define the interaction picture we split H as

$$H = H_0 + V$$

into a reference Hamiltonian H_0 and the Schrödinger picture potential $V := H - H_0$. We consider the reference evolution operator $U_0 = U_0(t)$ defined by the initial-value problem

$$i\hbar\dot{U}_0 = H_0U_0, \quad U_0(0) = 1,$$

and note that the solution U_0 is unitary,

$$U_0^* = U_0^{-1}.$$

In the interaction picture, the associated wave function is defined by

$$\psi_I := U_0^{-1}\psi = U_0^*\psi,$$

and operators X in the Schrödinger picture are represented by corresponding operators

$$X_I := U_0^{-1}XU_0 = U_0^*XU_0.$$

Note that $X \to X_I$ is an isomorphism:

$$(X \pm Y)_I = X_I \pm Y_I, \quad (XY)_I = X_IY_I, \quad (X^*)_I = (X_I)^*.$$

Since

$$\psi_I^*\psi_I = \psi^*U_0U_0^*\psi = \psi^*\psi,$$

states normalized in the Schrödinger picture are normalized in the interaction picture, and conversely, and for normalized states,

$$\langle X\rangle := \psi^*X\psi = (U_0\psi_I)^*XU_0\psi_I = \psi_I^*U_0^*XU_0\psi_I = \psi_I^*X_I\psi_I.$$

Thus quantum expectations are invariant under a change of the picture. From

$$\begin{aligned} i\hbar\dot{U}_0\psi_I + i\hbar U_0(\psi_I)^\bullet &= i\hbar(U_0\psi_I)^\bullet = i\hbar\dot{\psi} = H\psi = (H_0+V)U_0\psi_I \\ &= H_0U_0\psi_I + VU_0\psi_I = i\hbar\dot{U}_0\psi_I + U_0V_I\psi_I, \end{aligned}$$

we find $i\hbar U_0\dot{\psi}_I = U_0V_I\psi_I$, hence the dynamics

$$i\hbar(\psi_I)^\bullet = V_I\psi_I, \quad \psi_I(0) = \psi(0) \tag{1}$$ e.psiInt

for the wave function ψ_I in the interacting picture. Similarly

$$\begin{aligned} i\hbar(X_I)^\bullet &= i\hbar(U_0^*XU_0)^\bullet = -(i\hbar\dot{U}_0)^*XU_0 + U_0^*i\hbar\dot{X}U_0 + U_0^*Xi\hbar\dot{U}_0 \\ &= -(H_0U_0)^*XU_0 + U_0^*i\hbar\dot{X}U_0 + U_0^*XH_0U_0 = U_0^*(i\hbar\dot{X} - [H_0,X])U_0, \end{aligned}$$

hence

$$i\hbar(X_I)^\bullet = (i\hbar\dot{X} - [H_0,X])_I, \quad X_I(0) = X(0). \tag{2}$$ e.XIprop

2 A silver beam in the Stern–Gerlach experiment

s.silver

Now we apply this to quantum field theory in the halfspace $x_3 \geq 0$, modeling a beam of silver in the Stern–Gerlach experiment emanating from a hole in a plate placed at $x_3 = 0$. (At this stage of modeling there is no second plate where the silver would be absorbed and the measurement takes place. This will be remedied later.)

We use the second quantization formalism, writing $a(x)$ for time-independent fermionic 2-component annihilation operators, satisfying the canonical anticommutation relations

$$a_j(x)a_k(y) + a_k(y)a_j(x) = 0,$$

$$a_j(x)a_k(y)^* + a_k(y)^*a_j(x) = \delta_{jk}\delta(x-y).$$

$a(x)$ is the Fourier transform of the more conventional momentum space annihilation operators a_p. (The latter are convenient only in the absence of an external field.)

As reference Hamiltonian we take the 1-particle operator

$$H_0 := \int dx a(x)^* H_1(t,x,\widehat{p})a(x) \tag{3}$$ e.H0

where $H_1(t,x,\widehat{p})$ is the Hermitian single-particle Pauli Hamiltonian for a particle in an external magnetic field. To model the silver source behind the hole in the plate we need to add a Hermitian interaction term. 1-particle interactions require terms linear in $a(x)$ and $a(x)^*$, so we choose an interaction of the form

$$V = V(t) := a(J) + a(J)^*, \tag{4}$$ e.V

with the most general 1-particle annihilator

$$a(f) := \int dx J(t,x)a(x). \tag{5}$$

e.af

Since the Hamiltonian must be even, $J(t,x)$ must be an odd operator, acting on a Fermionic Fock space of hole states. Introducing time-independent fermionic 2-component hole annihilation operators b_h satisfying the canonical anticommutation relations

$$b_h b_{h'} + b_{h'} b_h = \delta_{hh'},$$

$$b_h a_k(y)^* + a_k(y)^* b_h = 0,$$

we take $J(t,x)$ to have the form

$$J(t,x) = \sum_h J_h(t,x) b_h,$$

where the $J_h(t,x)$ are complex 2×2 matrices with spinor indices, and a sum over spin indices is implied. The positions of the holes that are created in the silver source together with each silver atom in the beam need not be modeled explicitly since they are outside the halfspace $x_3 \geq 0$, hence are not affected by the dynamics on this halfspace.

The canonical anticommutation relations imply that H_0 commutes with the b_h and that

$$[H_0, a(J)] = -a(H_1 J), \quad [H_0, a(J)^*] = a(H_1 J)^*. \tag{6}$$

e.a1comm

Since the source producing the silver beam is outside the halfspace $x_3 \geq 0$, we must have

$$J_k(t,x) = 0 \quad \text{if } x_3 > 0. \tag{7}$$

e.Vboundary

Thus $J(t,x)$ only involves boundary terms at $x_3 = 0$. By (11), the dynamics (1) for the wave function in the interaction picture involves the interaction potential

$$V_I = a(J)_I + a(J)_I^* = a(U_1 J) + a_2(U_1 J)_I^*.$$

Thus

$$V_I = a(v) + a(v)^*, \tag{8}$$

e.VI

where

$$v := U_1 J = \sum_h v_h(t,x) b_h,$$

where the

$$v_h(t,x) := U_1 J_h(t,x)$$

satisfy the Pauli equation for a particle in an external magnetic field.