Preliminary version, December 7, 2022

1 The interaction picture

s.intpic

In the Schrödinger picture, a wave function $\psi = \psi(t)$ (time-dependence of wave functions and operators is suppressed, except in initial conditions) evolves according to the Schrödinger equation

$$i\hbar\dot{\psi} = H\psi,$$

where H is the Hamiltonian of the system of interest. To define the interaction picture we split H as

$$H = H_0 + V$$

into a reference Hamiltonian H_0 and the Schrödinger picture potential $V := H - H_0$. We consider the reference evolution operator $U_0 = U_0(t)$ defined by the initial-value problem

$$i\hbar\dot{U}_0 = H_0U_0, \quad U_0(0) = 1,$$

and note that the solution U_0 is unitary,

$$U_0^* = U_0^{-1}.$$

In the interaction picture, the associated wave function is defined by

$$\psi_I := U_0^{-1}\psi = U_0^*\psi,$$

and operators X in the Schrödinger picture are represented by corresponding operators

$$X_I := U_0^{-1}XU_0 = U_0^*XU_0.$$

Note that $X \to X_I$ is an isomorphism:

$$(X \pm Y)_I = X_I \pm Y_I, \quad (XY)_I = X_IY_I, \quad (X^*)_I = (X_I)^*.$$

Since

$$\psi_I^*\psi_I = \psi^*U_0U_0^*\psi = \psi^*\psi,$$

states normalized in the Schrödinger picture are normalized in the interaction picture, and conversely, and for normalized states,

$$\langle X\rangle := \psi^*X\psi = (U_0\psi_I)^*XU_0\psi_I = \psi_I^*U_0^*XU_0\psi_I = \psi_I^*X_I\psi_I.$$

Thus quantum expectations are invariant under a change of the picture. From

$$\begin{aligned} i\hbar\dot U_0\psi_I + i\hbar U_0(\psi_I)^\bullet &= i\hbar(U_0\psi_I)^\bullet = i\hbar\dot\psi = H\psi = (H_0+V)U_0\psi_I \\ &= H_0U_0\psi_I + VU_0\psi_I = i\hbar\dot U_0\psi_I + U_0V_I\psi_I, \end{aligned}$$

we find $i\hbar U_0\dot\psi_I = U_0V_I\psi_I$, hence the dynamics

$$i\hbar(\psi_I)^\bullet = V_I\psi_I, \quad \psi_I(0) = \psi(0) \tag{1}$$ e.psiInt

for the wave function ψ_I in the interacting picture. Similarly

$$\begin{aligned} i\hbar(X_I)^\bullet &= i\hbar(U_0^*XU_0)^\bullet = -(i\hbar\dot U_0)^*XU_0 + U_0^*i\hbar\dot XU_0 + U_0^*Xi\hbar\dot U_0 \\ &= -(H_0U_0)^*XU_0 + U_0^*i\hbar\dot XU_0 + U_0^*XH_0U_0 = U_0^*(i\hbar\dot X - [H_0,X])U_0, \end{aligned}$$

hence

$$i\hbar(X_I)^\bullet = (i\hbar\dot X - [H_0,X])_I, \quad X_I(0) = X(0). \tag{2}$$ e.XIprop

2 A silver beam in the Stern–Gerlach experiment

s.silver

Now we apply this to quantum field theory in the halfspace $x_3 \geq 0$, modeling a beam of silver in the Stern–Gerlach experiment emanating from a hole in a plate placed at $x_3 = 0$. We take as reference Hamiltonian the 1-particle operator

$$H_0 := \int dx a(x)^* H_1(t,x,\widehat p)a(x) \tag{3}$$ e.H0

where the $a(x)$ are time-independent fermionic 2-component creation and annihilation operators and $H_1(t,x,\widehat p)$ is the Hermitian single-particle Pauli Hamiltonian for a particle in an external magnetic field. The canonical anticommutation rules

$$a_j(x)a_k(y) + a_k(y)a_j(x) = 0,$$

$$a_j(x)a_k(y)^* + a_k(y)^*a_j(x) = \delta_{jk}\delta(x-y)$$

imply that a general 1-particle annihilator

$$a(f) := \int dx f(t,x)a(x)$$

with a 1-particle wave function $f = f(t,x)$ (and a sum over spin indices implied) satisfies

$$[H_0, a(f)] = -a(H_1f), \quad [H_0, a(f)^*] = a(H_1f)^*,$$

and that a general 2-particle annihilator

$$a_2(g) := \int dxdx' g(t,x,x')a(x)a(x')$$

with an antisymmetric 2-particle wave function $g(t, x, x')$ satisfies

$$[H_0, a_2(g)] = -a_2(H_2 g), \quad [H_0, a_2(g)^*] = a_2(H_2 g)^*, \tag{4}$$ e.H0comm

where $H_2 g$ denotes the sum of the two terms obtained by applying H_1 to the first and second argument of g. The functorial properties

$$a(f)_I = a(U_1 f), \quad a_2(g)_I = a_2(U_2 g), \tag{5}$$ e.aProp

where $U_2 g$ denotes the term obtained by applying U_1 first to the first and then to the second argument of g, can be proved easily by verifying with the help of (4) that the equations (2) hold. They express the fact that in the particle view of quantum field theory, particles move independently under a reference Hamiltonian of the form (3).

To model the silver source behind the plate we add a Hermitian interaction of the form (terms linear in $a(x)$ are forbidden since the Hamiltonian must be even)

$$V = V(t) := a_2(v) + a_2(v)^*,$$

where $v = v(t, x, x')$ is an antisymmetric 2-particle wave function with

$$v(t, x, x') = 0 \quad \text{for } x_3 > 0 \text{ or } x_3' > 0,$$

since the source producing the silver beam is outside the halfspace $x_3 \geq 0$. Thus v only involves boundary terms at $x_3 = 0$. By (5), the dynamics (1) for the wave function in the interaction picture involves the interaction potential

$$V_I = a_2(v)_I + a_2(v)_I^* = a_2(U_2 v) + a_2(U_2 v)_I^*.$$

Thus

$$V_I = a_2(w) + a_2(w)_I^*, \tag{6}$$ e.VI

where the 2-particle wave function

$$w := U_2 v$$

satisfies the 2-particle Schrödinger equation with the Hamiltonian

$$H_2 = H_1(t, x, \widehat{p}) + H_1(t, x', \widehat{p}')$$

for two independent particles in an external magnetic field.