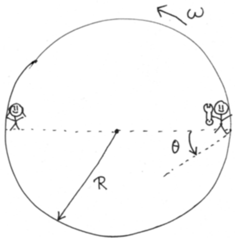


ω
θ
R

phermi

Our strategy for determining the launch angle θ will be to determine the paths of the wrench and of Sandra as viewed in the inertial frame. We compute these paths as a function of the launch angle θ of the wrench as viewed in the rotating frame. We then determine which launch angle(s) that cause the two paths to intersect at some time after the wrench is thrown. Since the wrench is not subjected to any forces after it is thrown, it will travel along a straight line path as viewed in the inertial frame. On the other hand, the path of the wrench as viewed in the disk frame will potentially be very complicated because that frame is non-inertial, so there are fictitious forces present in that frame. The first difficulty in solving the problem from the point of view of the inertial frame is that the initial data, namely the launch angle and launch speed, are given in the disk frame. Therefore, we first transform these data to the inertial frame. Let the cartesian coordinates as a function of time of the wrench as viewed by the inertial observer be denoted by

$$\begin{pmatrix} x_I(t) \\ y_I(t) \end{pmatrix},$$

and let the cartesian coordinates as a function of time of the wrench as viewed by an observer who is at rest relative to the disk be denoted by

$$\begin{pmatrix} x_D(t) \\ y_D(t) \end{pmatrix}.$$

For simplicity, we assume that the wrench is thrown at $t=0$, the "initial time" and that both observers choose the center of the disk as the origin of their coordinates. Moreover, we assume that the x-axes of both observers coincide at time $t=0$ and that the wrench is on the x-axis at the initial time. All of these remarks together imply the following initial conditions on the position of the wrench in both frames:

$$\begin{pmatrix} x_I(0) \\ y_I(0) \end{pmatrix} = \begin{pmatrix} R \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} x_D(0) \\ y_D(0) \end{pmatrix} = \begin{pmatrix} R \\ 0 \end{pmatrix}.$$

The problem statement also tells us that the wrench is thrown with speed $R\omega$ as viewed in the disk frame. This fact, combined with the definition of the launch angle $\theta $, gives the following additional initial data:

The problem statement also tells us that the wrench is thrown with speed $R\omega$ as viewed in the disk frame. This fact, combined with the definition of the launch angle θ, gives the following additional initial data:

$$\begin{pmatrix} \dot{x}_D(0) \\ \dot{y}_D(0) \end{pmatrix} = \begin{pmatrix} -R\omega\cos\theta \\ -R\omega\sin\theta \end{pmatrix}.$$

How can these data be used to determine the initial velocity of the wrench as viewed in the inertial frame? Recall that the disk is rotation counterclockwise with angular velocity ω as viewed from the inertial frame, so the inertial coordinates are related to the disk coordinates by a time-dependent, counterclockwise rotation.

$$\begin{pmatrix} x_I(t) \\ y_I(t) \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} x_D(t) \\ y_D(t) \end{pmatrix}$$

With a little calculus, one can differentiate both sides of this equation to relate the velocity of the wrench as viewed from the two frames. If one evaluates this result at time $t=0$, then one obtains

$$\begin{pmatrix} \dot{x}_I(0) \\ \dot{y}_I(0) \end{pmatrix} = \begin{pmatrix} \dot{x}_D(0) \\ \dot{y}_D(0) \end{pmatrix} + \omega \begin{pmatrix} -y_D(0) \\ x_D(0) \end{pmatrix} = \begin{pmatrix} -R\omega\cos\theta \\ R\omega(1-\sin\theta) \end{pmatrix}$$

Having determined the initial velocity of the wrench as viewed in the inertial frame, we use the fact that it travels along a straight line in the inertial frame to determine its path for all times as follows:

$$\begin{pmatrix} x_I(t) \\ y_I(t) \end{pmatrix} = \begin{pmatrix} x_I(0) + \dot{x}_I(0)t \\ y_I(0) + \dot{y}_I(0)t \end{pmatrix} = R \begin{pmatrix} 1-(\omega t)\cos\theta \\ (\omega t)(1-\sin\theta) \end{pmatrix}$$

On the other hand, Sandra's position as a function of time as measured by the inertial observer can be determined by noting that since at the initial time she is at the point diametrically opposite the wrench, her coordinates as measured in the disk frame are

$$\begin{pmatrix} X_D(t) \\ Y_D(t) \end{pmatrix} = \begin{pmatrix} -R \\ 0 \end{pmatrix}.$$

Applying the time-dependent rotation written above gives her inertial coordinates.

$$\begin{pmatrix} X_I(t) \\ Y_I(t) \end{pmatrix} = R \begin{pmatrix} -\cos(\omega t) \\ -\sin(\omega t) \end{pmatrix}.$$

In order for Sandra to catch the wrench at some time t, she must be at the same location as the wrench at that time, which means that her coordinates must match up with those of the wrench.

$$\begin{pmatrix} X_I(t) \\ Y_I(t) \end{pmatrix} = \begin{pmatrix} x_I(t) \\ y_I(t) \end{pmatrix}.$$

This gives two equations in two unknowns θ and ωt:

$$\begin{aligned} 1-(\omega t)\cos\theta &= -\cos(\omega t) \\ (\omega t)(1-\sin\theta) &= -\sin(\omega t). \end{aligned}$$

We want to solve these equations for all of the possible values of θ. If one solves for $\cos\theta$ and $\sin\theta$ respectively in these two equations, then one obtains

$$\begin{aligned} \cos\theta &= \frac{1+\cos(\omega t)}{\omega t} \\ \sin\theta &= 1+\frac{\sin(\omega t)}{\omega t}. \end{aligned}$$

By squaring both sides of each equation, adding the two resulting equations, and noting that $\sin^2\theta + \cos^2\theta =1 $, one obtains an equation that involves only $\omega t $.

$$1+(\omega t)\sin(\omega t)+\cos(\omega t) = 0.$$

One can now solve this equation to get a list of possible values for ωt, and then one can plug these values into either the equation for $\sin\theta$ or $\cos\theta$ and solve for θ to determine the corresponding possible values of θ. Inspection shows that if ωt is any odd integer multiple of π, then one obtains a solution to this equation, and any such value of ωt gives the following conditions on θ:

$$\sin\theta = 1, \qquad \cos\theta = 0.$$

This yields only one permissible θ, namely

$$\theta_\infty = \frac{\pi}{2}.$$

The reason for the "∞" subscript will become apparent in a moment. Numerical analysis reveals that there is an infinite sequence of solutions $\omega t_{\pm 1},\omega t_{\pm 2},\omega t_{\pm 3}\dots$ that are all close to even multiples of π and such that $\omega t_n\sim \pm 2n\pi$ as $n\to\infty$. However, when one plugs these into the equations for θ, one finds that only the positive solutions also satisfy those equations simultaneously. So one has the infinite sequence $\omega t_1, \omega t_2, \dots$ of solutons. The corresponding angles $\theta_1, \theta_2, \dots$ are obtained by plugging the ωt_n back into the equations for θ and solving for θ. The first ten smallest angles in radians accurate to the tenth decimal place are

$$\theta_1 = 1.2377836630$$

\dots$ of solutons. The corresponding angles $\theta_1, \theta_2, \dots$ are obtained by plugging the ωt_n back into the equations for θ and solving for θ. The first ten smallest angles in radians accurate to the tenth decimal place are

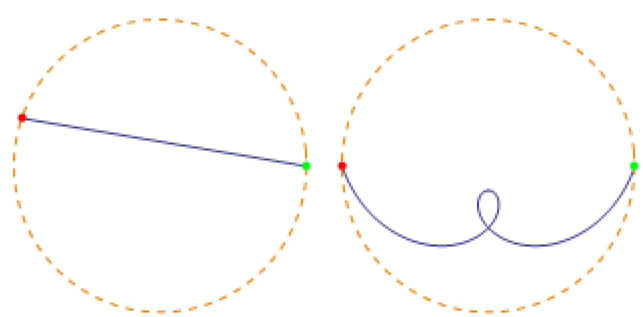
$$\begin{aligned}
\theta_1 &= 1.2377836630\\
\theta_2 &= 1.4099253460\\
\theta_3 &= 1.4641906170\\
\theta_4 &= 1.4910077734\\
\theta_5 &= 1.5070264813\\
\theta_6 &= 1.5176823197\\
\theta_7 &= 1.5252842403\\
\theta_8 &= 1.5309813101\\
\theta_9 &= 1.5354101088\\
\theta_{10} &= 1.5389518887
\end{aligned}$$

It is not hard to see numerically that these solutions converge to $\pi/2$, namely

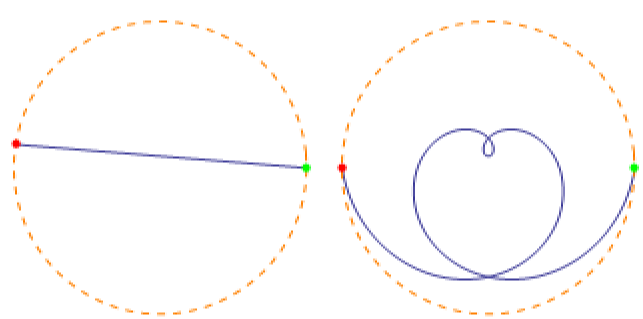
$$\lim_{n\to\infty} \theta_n = \frac{\pi}{2}.$$

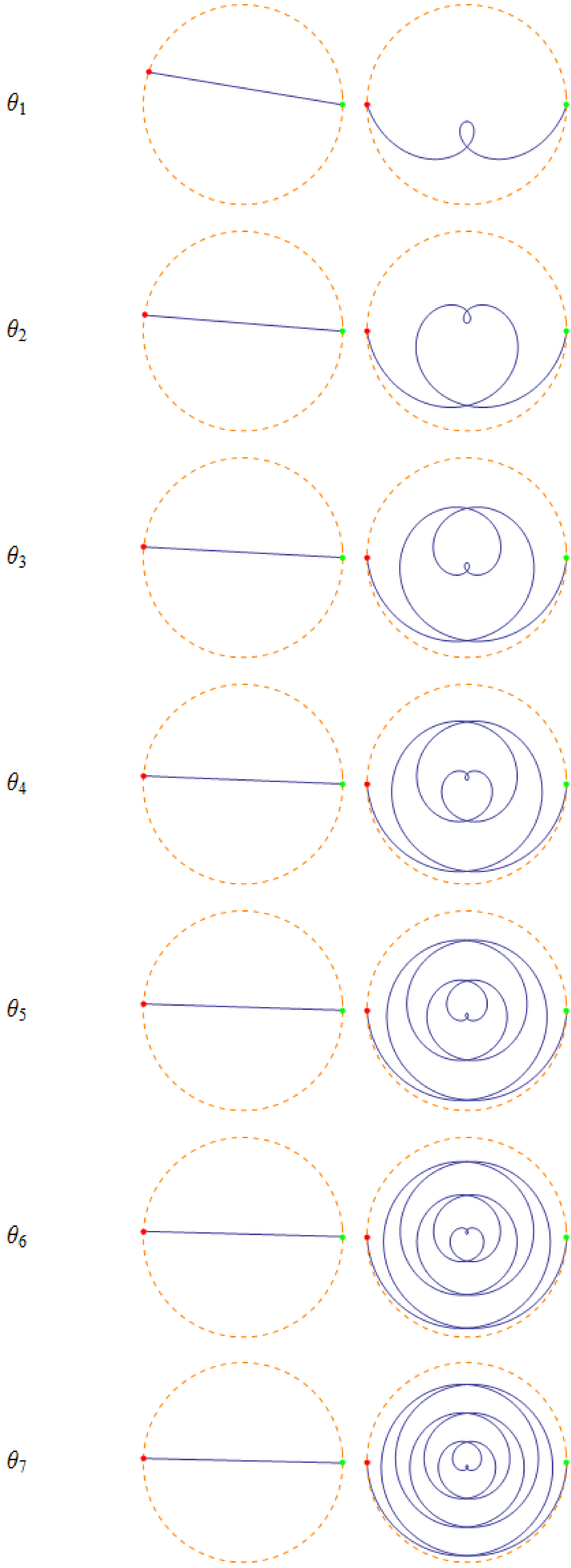
This is why we chose the θ_∞ notation from earlier for the solution $\theta = \pi/2$. Below, we have plotted the trajectory of the wrench in both the inertial and rotating frame for the smallest seven launch angles. In each row, the left-hand plot displays the situation as viewed from the inertial frame, and the right-hand plot displays the situation as viewed from the rotating frame. The green dot is the position from which the wrench is thrown, and the red dot is the position at which it is caught. Notice that as viewed in the rotating frame, the curve corresponding to launch angle θ_n intersects itself exactly n times. Can you explain why this is happening and what it means physically?

θ_1



θ_2




θ_1
θ_2
θ_3
θ_4
θ_5
θ_6
θ_7