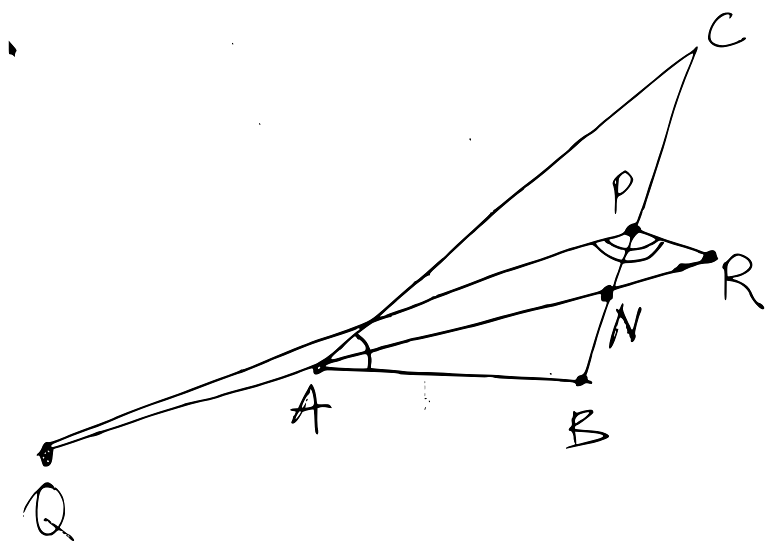

There are two triangles ABC and PQR. The vertex A is a middle of the side QR. The vertex P is a middle of the side BC. The line QR is a bisector of the angle BAC. The line BC is a bisector of the angle QPR. Prove that |AC|+|AB|=|PR|+|PQ|.

First we solve this problem under the following additional assumption: there are no parallel lines among AB, AC, PR, PQ.

Observe that

$$\boldsymbol{AP} = \frac{1}{2}(\boldsymbol{AB} + \boldsymbol{AC}) = -\frac{1}{2}(\boldsymbol{PQ} + \boldsymbol{PR}) = \boldsymbol{AN} - \boldsymbol{PN}, \tag{1}$$

where the bisectors are presented as follows

$$\boldsymbol{AN} = \frac{|AB|\boldsymbol{AC}}{|AB| + |AC|} + \frac{|AC|\boldsymbol{AB}}{|AB| + |AC|}, \quad \boldsymbol{PN} = \frac{|PQ|\boldsymbol{PR}}{|PQ| + |PR|} + \frac{|PR|\boldsymbol{PQ}}{|PQ| + |PR|}.$$

Equations (1) imply

$$\boldsymbol{PQ} = \frac{(c+b)\boldsymbol{AC} + (b-c)\boldsymbol{AB}}{a-b}, \tag{2}$$

$$\boldsymbol{PR} = -\frac{(c+a)\boldsymbol{AC} + (a-c)\boldsymbol{AB}}{a-b}, \tag{3}$$

where

$$a = \frac{|PR|}{|PR| + |PQ|}, \quad b = \frac{|PQ|}{|PR| + |PQ|}, \quad c = \frac{|AB| - |AC|}{2(|AB| + |AC|)}. \tag{4}$$

Equality $a = b$ is impossible. Indeed, if it would be so then $|PR| = |PQ|$ and PQR is an isosceles triangle so that $A = N$ and the triangle ABC collapses; by the analogous reason $c \neq 0$.

Formulas (2), (3) imply respectively

$$\begin{aligned} |PQ|^2(a-b)^2 &= (c+b)^2|AC|^2 + (b-c)^2|AB|^2 + 2(c+b)(b-c)S, \\ |PR|^2(a-b)^2 &= (c+a)^2|AC|^2 + (a-c)^2|AB|^2 + 2(c+a)(a-c)S, \end{aligned} \tag{5}$$

where $S = (\boldsymbol{AC}, \boldsymbol{AB})$.

Excluding $2S$ from equations (5) we obtain

$$\begin{aligned}&\frac{|PQ|^2(a-b)^2-(c+b)^2|AC|^2-(b-c)^2|AB|^2}{(c+b)(b-c)}\\&-\frac{|PR|^2(a-b)^2-(c+a)^2|AC|^2-(a-c)^2|AB|^2}{(c+a)(a-c)}=0.\end{aligned}\tag{6}$$

The denominators in equation (6) cannot vanish provided there are no parallel lines among AB, AC, PR, PQ. Indeed, if for example $b-c=0$ then by equation (2) PQ is parallel to AC etc.

The left side of (6) is processed with the help of Maple. See the sheet below.

Without loss of generality we can put $|AB|>|AC|$. Indeed, else one must replace the notations for vertexes A and C. Analogously the assumption $|PQ|>|PR|$ does not lose the generality. These inequalities imply $b>a, \quad c>0$.

Consider tha case $a=c$. From (3) it follows that

$$|\boldsymbol{PR}|=\frac{(c+a)|\boldsymbol{AC}|}{b-a}=\frac{2a|\boldsymbol{AC}|}{b-a}=|\boldsymbol{AC}|\frac{2|PR|}{|PQ|-|PR|};$$

or

$$|PQ|-|PR|=2|AC|.$$

This equation together with $a=c$ (see (4) gives

$$|PQ|=\frac{1}{2}|AB|+\frac{3}{2}|AC|,\quad |PR|=\frac{1}{2}(|AB|-|AC|);$$

and formula

$$|PQ|+|PR|=|AB|+|AC|$$

holds.

$> a := \frac{PR}{PR+PQ}$

$$a := \frac{PR}{PR+PQ} \quad \textbf{(1)}$$

$> b := \frac{PQ}{PR+PQ}$

$$b := \frac{PQ}{PR+PQ} \quad \textbf{(2)}$$

$> c := \frac{(AB-AC)}{2\cdot(AB+AC)}$

$$c := \frac{AB-AC}{2\,AB+2\,AC} \quad \textbf{(3)}$$

$>$

$> U := \frac{\left(PQ^2\cdot(a-b)^2-(c+b)^2\cdot AC^2-(b-c)^2\cdot AB^2\right)}{b^2-c^2}$

$$U := \frac{1}{\frac{PQ^2}{(PR+PQ)^2}-\frac{(AB-AC)^2}{(2\,AB+2\,AC)^2}}\left(PQ^2\left(\frac{PR}{PR+PQ}-\frac{PQ}{PR+PQ}\right)^2-\left(\frac{AB-AC}{2\,AB+2\,AC}+\frac{PQ}{PR+PQ}\right)^2 AC^2-\left(\frac{PQ}{PR+PQ}-\frac{AB-AC}{2\,AB+2\,AC}\right)^2 AB^2\right) \quad \textbf{(4)}$$

$> V := \frac{\left(PR^2\cdot(a-b)^2-(c+a)^2\cdot AC^2-(a-c)^2\cdot AB^2\right)}{a^2-c^2}$

$$V := \frac{1}{\frac{PR^2}{(PR+PQ)^2}-\frac{(AB-AC)^2}{(2\,AB+2\,AC)^2}}\left(PR^2\left(\frac{PR}{PR+PQ}-\frac{PQ}{PR+PQ}\right)^2-\left(\frac{AB-AC}{2\,AB+2\,AC}+\frac{PR}{PR+PQ}\right)^2 AC^2-\left(\frac{PR}{PR+PQ}-\frac{AB-AC}{2\,AB+2\,AC}\right)^2 AB^2\right) \quad \textbf{(5)}$$

$> W := U-V$

$$W := \frac{1}{\frac{PQ^2}{(PR+PQ)^2}-\frac{(AB-AC)^2}{(2\,AB+2\,AC)^2}}\left(PQ^2\left(\frac{PR}{PR+PQ}-\frac{PQ}{PR+PQ}\right)^2-\left(\frac{AB-AC}{2\,AB+2\,AC}+\frac{PQ}{PR+PQ}\right)^2 AC^2-\left(\frac{PQ}{PR+PQ}-\frac{AB-AC}{2\,AB+2\,AC}\right)^2 AB^2\right) - \frac{1}{\frac{PR^2}{(PR+PQ)^2}-\frac{(AB-AC)^2}{(2\,AB+2\,AC)^2}}\left(PR^2\left(\frac{PR}{PR+PQ}-\frac{PQ}{PR+PQ}\right)^2-\left(\frac{AB-AC}{2\,AB+2\,AC}+\frac{PR}{PR+PQ}\right)^2 AC^2-\left(\frac{PR}{PR+PQ}-\frac{AB-AC}{2\,AB+2\,AC}\right)^2 AB^2\right) \quad \textbf{(6)}$$

$> factor(W)$

(7)

$$-\left(4\,(PR+PQ)\,(-PR+PQ)^3\,(AB-AC)^2\,(AB+AC)^2\,\underline{(AC-PQ-PR+AB)}\,(AC+AB+PQ+PR)\right)\Big/\left((-AB\,PR-3\,AC\,PR+AB\,PQ-AC\,PQ)\,(3\,AB\,PR+AB\,PQ+AC\,PR-AC\,PQ)\,(AB\,PQ+3\,AC\,PQ-AB\,PR+AC\,PR)\,(AB\,PR+3\,AB\,PQ-AC\,PR+AC\,PQ)\right) \tag{7}$$

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