APPLIED FINITE ELEMENT ANALYSIS

Second Edition

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To my parents T. J. and Bessie and my grade school teacher Alice Brunger

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The above equations are equivalent to

$$\frac{\partial(\{U\}^T[A]\{U\})}{\partial\{U\}}=[A]\{U\}+[A]^T\{U\} \tag{10}$$

When $[A]$ is symmetric,

$$\frac{\partial(\{U\}^T[A]\{U\})}{\partial\{U\}}=2[A]\{U\} \tag{11}$$

Appendix 3

MODIFYING THE SYSTEM OF EQUATIONS

The system of equations

$$[K]\{\Phi\}=\{F\}$$

or

$$[K]\{U\}=\{F\}+\{P\}$$

obtained by using the direct stiffness procedure must be modified whenever some of the values in $\{\Phi\}$ or $\{U\}$ are known. All field problems except some problems involving convection heat transfer must have some of the boundary values specified and all solid mechanics problems must have displacements specified to eliminate rigid body motion. Therefore, the modification of the system of equations to incorporate known nodal conditions is more the rule than the exception.

Our objective here is to discuss and then illustrate a systematic procedure for modifying $[K]$ and $\{F\}$ such that we satisfy two criteria. First, we must obtain the correct answers for all values in $\{\Phi\}$ or $\{U\}$. Second, we do not want to change the size of $[K]$, $\{F\}$, and $\{P\}$ because this leads to programming difficulties. We shall consider the steady-state situation first and then discuss the modification of equations associated with time-dependent field problems.

III.1 STEADY-STATE EQUATIONS

The modification of the system of equations $[K]\{\Phi\}=\{F\}$ is a two-step procedure once the subscript of the known nodal parameter is available. For example, suppose that Φ_5 has a known value. The modification proceeds as follows.

1. All of the coefficients in row five are set equal to zero except the diagonal term, which is left unaltered. In equation form, $K_{5j}=0, j=1,\ldots,n$ and $j\neq 5$. The associated term in the column vector $\{F\}$, F_5, is replaced by the product $K_{55}\Phi_5$.
2. All of the remaining equations are modified by subtracting the product $K_{j5}\Phi_5$ from F_j and then setting $K_{j5}=0, j=1,\ldots,n, j\neq 5$.

ILLUSTRATIVE EXAMPLE

Modify the following system of equations when $\Phi_1 = 150$ and $\Phi_5 = 40$.

$$\begin{bmatrix} 55 & -46 & 0 & 0 & 0 \\ -46 & 140 & -46 & 0 & 0 \\ 4 & -46 & 110 & -46 & 4 \\ 0 & 0 & -46 & 142 & -46 \\ 0 & 0 & 4 & -46 & 65 \end{bmatrix} \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \Phi_5 \end{Bmatrix} = \begin{Bmatrix} 500 \\ 2000 \\ 1000 \\ 2000 \\ 900 \end{Bmatrix}$$

To implement step one, we set all of the coefficients in rows one and five to zero except the diagonal terms, which are left unaltered. The corresponding terms in $\{F\}$, F_1 and F_5, are then replaced by $F_1 = K_{11}\Phi_1$ and $F_5 = K_{55}\Phi_5$, respectively. This step yields

$$\begin{bmatrix} 55 & 0 & 0 & 0 & 0 \\ -46 & 140 & -46 & 0 & 0 \\ 4 & -46 & 110 & -46 & 4 \\ 0 & 0 & -46 & 142 & -46 \\ 0 & 0 & 0 & 0 & 65 \end{bmatrix} \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \Phi_5 \end{Bmatrix} = \begin{Bmatrix} 8250 \\ 2000 \\ 1000 \\ 2000 \\ 2600 \end{Bmatrix}$$

The second step involves the elimination of the columns of coefficients that multiply Φ_1 and Φ_5. This is accomplished by transferring the coefficients involving Φ_1 and Φ_5 to the right-hand side. For example, F_2 becomes $2000 + 46\Phi_1$, or 8900. Completion of this step gives

$$\begin{bmatrix} 55 & 0 & 0 & 0 & 0 \\ 0 & 140 & -46 & 0 & 0 \\ 0 & -46 & 110 & -46 & 0 \\ 0 & 0 & -46 & 142 & 0 \\ 0 & 0 & 0 & 0 & 65 \end{bmatrix} \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \Phi_5 \end{Bmatrix} = \begin{Bmatrix} 8250 \\ 8900 \\ 240 \\ 3840 \\ 2600 \end{Bmatrix}$$

III.2 TIME-DEPENDENT EQUATIONS

The incorporation of specified nodal values in time-dependent problems is more complicated because the solution procedure involves combinations of $[C]$ and $[K]$, namely $[A]$ and $[P]$. We shall place the same requirement on the time-dependent solution that was placed on the steady-state solution. We want to keep the dimensions of $[C]$ and $[K]$ and thus $[A]$ and $[P]$ the same after modification as they were before modification.

The algorithm for modifying $[C]$ and $[K]$ is more easily understood once we have looked at a specific problem. Let us reconsider the problem in Section 14.5 without the heat source at node one. Instead we assume that $\Phi_1 = 40°\text{C}$ for all time values. The vector of initial conditions $\{\Phi\}_a$ becomes $\{\Phi\}_a^T = [40 \quad 0 \quad 0]$.

Our desire is to maintain $[A]$ and $[P]$ as 3×3 matrices with the new property that $\Phi_1 = 40°\text{C}$ for all of the calculated solutions.

If we use a lumped capacitance matrix with the $[K]$ that was obtained for the example problem in Section 14.5, the system of differential equations is

$$\begin{aligned} 12\frac{d\Phi_1}{dt} + 2\Phi_1 - 2\Phi_2 &= 0 \\ 24\frac{d\Phi_2}{dt} - 2\Phi_1 + 4\Phi_2 - 2\Phi_3 &= 0 \\ 12\frac{d\Phi_3}{dt} - 2\Phi_2 + 2\Phi_3 &= 0 \end{aligned} \tag{1}$$

The first equation of (1) comes from $R_1 = 0$. Since Φ_1 has the fixed value of 40, the first equation should not be included. We must eliminate the residual equation for node one because Φ_1 is known. The correct system of differential equations is

$$\begin{aligned} 24\frac{d\Phi_2}{dt} + 4\Phi_2 - 2\Phi_3 - 80 &= 0 \\ 12\frac{d\Phi_3}{dt} - 2\Phi_2 + 2\Phi_3 &= 0 \end{aligned} \tag{2}$$

The value -80 in the first equation of (2) comes from substituting $\Phi_1 = 40$ into $-2\Phi_1$ in the original equation.

The equivalent matrix form is

$$\begin{bmatrix} 24 & 0 \\ 0 & 12 \end{bmatrix} \frac{d\{\Phi^*\}}{dt} + \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \{\Phi^*\} - \begin{Bmatrix} 80 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3}$$

where $\{\Phi^*\}^T = [\Phi_2 \quad \Phi_3]$.

A central difference solution of (3) using $\Delta t = 1$ is

$$\begin{bmatrix} 26 & -1 \\ -1 & 13 \end{bmatrix} \{\Phi\}_b = \begin{bmatrix} 22 & -1 \\ -1 & 11 \end{bmatrix} \{\Phi\}_a + \begin{Bmatrix} 80 \\ 0 \end{Bmatrix} \tag{4}$$

Our objective is to have a system of equations that includes (4) but which also gives the correct value of Φ_1 for each time step. One way of achieving this objective is to expand (4) into a larger system as follows.

$$\begin{bmatrix} 12 & 0 & 0 \\ 0 & 26 & -1 \\ 0 & -1 & 13 \end{bmatrix} \{\Phi\}_b = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 22 & -1 \\ 0 & -1 & 11 \end{bmatrix} \{\Phi\}_a + \begin{Bmatrix} 0 \\ 80 \\ 0 \end{Bmatrix} \tag{5}$$

The diagonal values A_{11} and P_{11} are set equal to C_{11}. All of the other coefficients in the first row of $[A]$ and $[P]$ are zero. When (5) is solved, Φ_1 at $t = b$ will be the same as Φ_1 at $t = a$.

The modification of a system of differential equations when some of the nodal values are known can be accomplished by the following algorithm. Assume that

Φ_i is the known nodal value.

1. Add the products $K_{ji}\Phi_i$, $j=1,\ldots,n$ to the corresponding coefficient in $\{F\}$, that is, F_j.
2. Replace the coefficients in row i and column i of $[K]$ by zeros.
3. Set $F_i=0$.
4. If $[C]$ comes from the consistent formulation, sum the coefficients in row i and replace C_{ii} with this sum. Set all of the off-diagonal coefficients in the row to zero.

When these three steps are completed, $[A]$ and $[P]$ have properties similar to the matrices in (6). The diagonal coefficients, A_{ii} and P_{ii}, are the same as C_{ii}; thus, Φ_i at time b is the same as Φ_i at time a.

Answers to Selected Problems

1.1 $y=-\dfrac{4wH^4}{\pi^5 EI}\sin\dfrac{\pi x}{H}$

1.3 $y=-\dfrac{0.01326wH^4}{EI}\sin\dfrac{\pi x}{H}$

1.6 $y=-\dfrac{2M_0H^2}{\pi^3 EI}\sin\dfrac{\pi x}{H}$

1.9 $y=-\dfrac{0.0645M_0H^2}{EI}\sin\dfrac{\pi x}{H}$

1.11 $y=-\dfrac{PH^3}{16\pi EI}\sin\dfrac{\pi x}{H}$

1.14 $y=-\dfrac{4M_0H^2}{\pi^3 EI}$

$$\times\left(\sin\frac{\pi x}{H}+\frac{1}{27}\sin\frac{3\pi x}{H}\right)$$

1.16 $x/H=0.3008$

2.1 (a) 50.9, (c) 55.8

2.2 (a) -11.3, (c) -12.0

2.5 $N_i(s)=1-\dfrac{s}{L}$, $N_j(s)=\dfrac{s}{L}$

2.8 $N_i=\left(1-\dfrac{s}{L_s}\right)\left(1-\dfrac{t}{L_t}\right)$,

$$N_j=\frac{s}{L_s}\left(1-\frac{t}{L_t}\right),\ N_k=\frac{st}{L_sL_t},$$

$$N_m=\frac{t}{L_t}\left(1-\frac{s}{L_s}\right)$$

3.2 $\Phi_2=19.1$, $\Phi_3=-4.48$

3.4 (a) $\Phi_2=2.25$, $\Phi_3=3.50$, $\Phi_4=3.75$

3.4 (c) $\Phi_2=0.50$, $\Phi_3=-0.25$, $\Phi_4=-0.25$

3.8 $Y_2=-7.33$, $Y_3=-10.67$, $Y_4=-7.33$

5.5 $N_i+N_j+\cdots+N_m=1$

5.7 (a) $\phi=171.7$, (b) (0.21, 0.04), (0.202,0.088),

(c) $\dfrac{\partial\phi}{\partial x}=-230.8$, $\dfrac{\partial\phi}{\partial y}=-38.41$

5.9 (a) $\phi=170.9$, (b) (0.183, 0.13), (0.166, 0.166),

(c) $\dfrac{\partial\phi}{\partial x}=-283.3$, $\dfrac{\partial\phi}{\partial y}=-133.3$

5.12 (a) $\phi=92.47$, (b) (0.368, 0.18), (0.346, 0.25),

(c) $\dfrac{\partial\phi}{\partial x}=-420.4$, $\dfrac{\partial\phi}{\partial y}=-134.7$

5.14 (a) $\phi=140.3$, (b) (0.259, 0.060), (0.250, 0.095),

(c) $\dfrac{\partial\phi}{\partial x}=-438.1$, $\dfrac{\partial\phi}{\partial y}=-92.9$

6.4 $N_i=\dfrac{2}{3}-\dfrac{r}{L}$, $N_j=\dfrac{1}{3}+\dfrac{r}{L}$