Scalar-scalar (neutral) interaction

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1 Lagrangian

I consider the Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1\partial^\mu\phi_1 - m_1^2\phi_1^2) + \frac{1}{2}(\partial_\mu\phi_2\partial^\mu\phi_2 - m_2^2\phi_2^2) + \lambda\phi_1\phi_2$$

which gives the following Euler-Lagrange equations:

$$(\Box + m_1^2)\phi_1 = \lambda\phi_2$$
$$(\Box + m_2^2)\phi_2 = \lambda\phi_1$$

2 Canonical quantization

Since the interaction Lagrangian doesn't contain derivatives of the fields one has, as in the free case, $\pi_1 = \partial_0\phi_1$ and $\pi_2 = \partial_0\phi_2$. Then one expands the fields (a thing one can always do) like this:

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\phi_1(t,\boldsymbol{p})e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$$
$$\psi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\phi_2(t,\boldsymbol{p})e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$$

with $\phi_1^\dagger(t,\boldsymbol{p}) = \phi_1(t,-\boldsymbol{p})$ and $\phi_2^\dagger(t,\boldsymbol{p}) = \phi_2(t,-\boldsymbol{p})$ so that both ϕ_1 and ϕ_2 are hermitian. Then, since canonical quantization demands that canonical commutation relations hold between fields and their conjugate momenta, these relations must hold:[1]

$$[\phi_i(t_0,\boldsymbol{p}), \phi_j(t_0,\boldsymbol{k})] = 0 \ \forall\, \boldsymbol{p}, \boldsymbol{k}, i, j$$
$$[\dot\phi_i(t_0,\boldsymbol{p}), \dot\phi_j(t_0,\boldsymbol{k})] = 0 \ \forall\, \boldsymbol{p}, \boldsymbol{k}, i, j$$
$$[\phi_i(t_0,\boldsymbol{p}), \dot\phi_j(t_0,\boldsymbol{k})] = i\delta_{ij}\delta(\boldsymbol{p}-\boldsymbol{k})$$

[1] I've taken them at time t_0 because from Noether's theorem I know that there is an Hamiltonian such that $\phi_j(t,\boldsymbol{x}) = e^{iHt}\phi_j(0,\boldsymbol{x})e^{-iHt}$. Then these relations will continue to hold at a time different from t_0.

The first problem one gets into is the fact that it should be $i\delta(\boldsymbol{p}+\boldsymbol{k})$ in the second relation above, in order to get a $i\delta(\boldsymbol{x}-\boldsymbol{y})$. Let's see how far I can get before having to refer to that again. First of all, the equations one has to solve are:

$$\begin{aligned}
\ddot{\phi}_1(t,\boldsymbol{p}) &= -\omega_1^2(\boldsymbol{p})\phi_1(t,\boldsymbol{p}) + \lambda\phi_2(t,\boldsymbol{p}) \\
\ddot{\phi}_2(t,\boldsymbol{p}) &= -\omega_2^2(\boldsymbol{p})\phi_2(t,\boldsymbol{p}) + \lambda\phi_1(t,\boldsymbol{p})
\end{aligned}$$

then one calls $\dot{\phi}_1(t,\boldsymbol{p}) = \pi_1(t,\boldsymbol{p})$. There comes again the problem that one should have:

$$[\phi_1(t_0,\boldsymbol{p}), \pi_1(t_0,\boldsymbol{k})] = i\delta(\boldsymbol{p}+\boldsymbol{k})$$

so the reasoning I'm going to make to solve the equations doesn't convince me so much. Same problem for $\dot{\phi}_2(t,\boldsymbol{p}) = \pi_2(t,\boldsymbol{p})$.

Then, the only thing left is to solve quantum mechanically the equations:

$$\begin{aligned}
\dot{\phi}_1(t,\boldsymbol{p}) &= \pi_1(t,\boldsymbol{p}) \\
\dot{\phi}_2(t,\boldsymbol{p}) &= \pi_2(t,\boldsymbol{p}) \\
\dot{\pi}_1(t,\boldsymbol{p}) &= -\omega_1^2(\boldsymbol{p})\phi_1(t,\boldsymbol{p}) + \lambda\phi_2(t,\boldsymbol{p}) \\
\dot{\pi}_2(t,\boldsymbol{p}) &= -\omega_2^2(\boldsymbol{p})\phi_2(t,\boldsymbol{p}) + \lambda\phi_1(t,\boldsymbol{p})
\end{aligned}$$

that is:

$$\begin{pmatrix} \dot{\phi}_1(t,\boldsymbol{p}) \\ \dot{\phi}_2(t,\boldsymbol{p}) \\ \dot{\pi}_1(t,\boldsymbol{p}) \\ \dot{\pi}_2(t,\boldsymbol{p}) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\omega_1^2(\boldsymbol{p}) & \lambda & 0 & 0 \\ \lambda & -\omega_2^2(\boldsymbol{p}) & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1(t,\boldsymbol{p}) \\ \phi_2(t,\boldsymbol{p}) \\ \pi_1(t,\boldsymbol{p}) \\ \pi_2(t,\boldsymbol{p}) \end{pmatrix}$$

Forgetting the problem I spoke of before (because after all it arises in the free case, too. But there things come off pretty well, so why bother?), I asked Mathematica to diagonalize the matrix with $m_1 = m_2$ (otherwise it is a nightmare). One gets that (with $\omega_1^2(\boldsymbol{p}) = \omega_2^2(\boldsymbol{p}) = \omega^2(\boldsymbol{p})$):

$$\begin{pmatrix} \dot{a}_-(t,\boldsymbol{p}) \\ \dot{a}_+(t,\boldsymbol{p}) \\ \dot{b}_-(t,\boldsymbol{p}) \\ \dot{b}_+(t,\boldsymbol{p}) \end{pmatrix} = \begin{pmatrix} -i\sqrt{\omega^2(\boldsymbol{p})+\lambda} & 0 & 0 & 0 \\ 0 & i\sqrt{\omega^2(\boldsymbol{p})+\lambda} & 0 & 0 \\ 0 & 0 & -i\sqrt{\omega^2(\boldsymbol{p})-\lambda} & 0 \\ 0 & 0 & 0 & i\sqrt{\omega^2(\boldsymbol{p})-\lambda} \end{pmatrix} \begin{pmatrix} a_-(t,\boldsymbol{p}) \\ a_+(t,\boldsymbol{p}) \\ b_-(t,\boldsymbol{p}) \\ b_+(t,\boldsymbol{p}) \end{pmatrix}$$

with:

$$\begin{pmatrix} a_-(t,\boldsymbol{p}) \\ a_+(t,\boldsymbol{p}) \\ b_-(t,\boldsymbol{p}) \\ b_+(t,\boldsymbol{p}) \end{pmatrix} = \begin{pmatrix} \frac{i\sqrt{\omega^2(\boldsymbol{p})+\lambda}}{4} & -\frac{i\sqrt{\omega^2(\boldsymbol{p})+\lambda}}{4} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{i\sqrt{\omega^2(\boldsymbol{p})+\lambda}}{4} & \frac{i\sqrt{\omega^2(\boldsymbol{p})+\lambda}}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{i\sqrt{\omega^2(\boldsymbol{p})-\lambda}}{4} & -\frac{i\sqrt{\omega^2(\boldsymbol{p})-\lambda}}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{i\sqrt{\omega^2(\boldsymbol{p})-\lambda}}{4} & \frac{i\sqrt{\omega^2(\boldsymbol{p})-\lambda}}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \phi_1(t,\boldsymbol{p}) \\ \phi_2(t,\boldsymbol{p}) \\ \pi_1(t,\boldsymbol{p}) \\ \pi_2(t,\boldsymbol{p}) \end{pmatrix}$$

and:

$$\begin{pmatrix}\phi_1(t,\boldsymbol{p})\\ \phi_2(t,\boldsymbol{p})\\ \pi_1(t,\boldsymbol{p})\\ \pi_2(t,\boldsymbol{p})\end{pmatrix}=\begin{pmatrix}-\frac{i}{\sqrt{\omega^2(\boldsymbol{p})+\lambda}} & \frac{i}{\sqrt{\omega^2(\boldsymbol{p})+\lambda}} & \frac{i}{\sqrt{\omega^2(\boldsymbol{p})-\lambda}} & -\frac{i}{\sqrt{\omega^2(\boldsymbol{p})-\lambda}}\\ \frac{i}{\sqrt{\omega^2(\boldsymbol{p})+\lambda}} & -\frac{i}{\sqrt{\omega^2(\boldsymbol{p})+\lambda}} & \frac{i}{\sqrt{\omega^2(\boldsymbol{p})-\lambda}} & -\frac{i}{\sqrt{\omega^2(\boldsymbol{p})-\lambda}}\\ -1 & -1 & 1 & 1\\ 1 & 1 & 1 & 1\end{pmatrix}\begin{pmatrix}a_-(t,\boldsymbol{p})\\ a_+(t,\boldsymbol{p})\\ b_-(t,\boldsymbol{p})\\ b_+(t,\boldsymbol{p})\end{pmatrix}$$

3 Vacuum and one particle states

First of all I noticed that in order to have that the eigenvalues of the diagonal matrix that dictates the evolution of the a and b are purely imaginary (as it should be in quantum mechanics), it must be that $m^2-\lambda\geqslant 0$ and $m^2+\lambda\geqslant 0$.

This is because $\omega^2(\boldsymbol{p})-\lambda=m^2+\boldsymbol{p}^2-\lambda$ and $\omega^2(\boldsymbol{p})+\lambda=m^2+\boldsymbol{p}^2+\lambda$, then since $\boldsymbol{p}^2$ is at least 0, it sure is $\omega^2(\boldsymbol{p})-\lambda\geqslant 0$ and $\omega^2(\boldsymbol{p})+\lambda\geqslant 0$ if $m^2-\lambda\geqslant 0$ and $m^2+\lambda\geqslant 0$. The condition on λ is:

$$|\lambda|\leqslant m^2$$

For simplicity I'll call $\sqrt{\omega^2(\boldsymbol{p})+\lambda}=\omega_a(\boldsymbol{p})>0$ and $\sqrt{\omega^2(\boldsymbol{p})-\lambda}=\omega_b^2(\boldsymbol{p})>0$. With that said, since one has that there exists (from Noether theorem) a $P^\mu=(H,\boldsymbol{P})$ operator conserved in time that generates spacetime translations, it must be that:

$$\begin{aligned}\dot{a}_-(t,\boldsymbol{p})&=-i[a_-(t,\boldsymbol{p}),H]=-i\omega_a(\boldsymbol{p})a_-(t,\boldsymbol{p})\\ \dot{a}_+(t,\boldsymbol{p})&=-i[a_+(t,\boldsymbol{p}),H]=i\omega_a(\boldsymbol{p})a_+(t,\boldsymbol{p})\\ \dot{b}_-(t,\boldsymbol{p})&=-i[b_-(t,\boldsymbol{p}),H]=-i\omega_b(\boldsymbol{p})b_-(t,\boldsymbol{p})\\ \dot{b}_+(t,\boldsymbol{p})&=-i[b_+(t,\boldsymbol{p}),H]=i\omega_b(\boldsymbol{p})b_+(t,\boldsymbol{p})\end{aligned}$$

and then:

$$\begin{aligned}[][H,a_-(t,\boldsymbol{p})]&=-\omega_a(\boldsymbol{p})a_-(t,\boldsymbol{p})\\ [H,a_+(t,\boldsymbol{p})]&=\omega_a(\boldsymbol{p})a_+(t,\boldsymbol{p})\\ [H,b_-(t,\boldsymbol{p})]&=-\omega_b(\boldsymbol{p})b_-(t,\boldsymbol{p})\\ [H,b_+(t,\boldsymbol{p})]&=\omega_b(\boldsymbol{p})b_+(t,\boldsymbol{p})\end{aligned}$$

After this observation, I saw that since the Hamiltonian is:

$$\left(\int \mathrm{d}^3x\sum_{j=1}^{2}\{\pi_j(t_0,\boldsymbol{x})^2+\|\nabla\phi_j(t_0,\boldsymbol{x})\|^2\}\right)+\left(\int \mathrm{d}^3x\left\{\frac{m^2}{2}(\phi_1^2+\phi_2^2)-\lambda\phi_1\phi_2\right\}\right)=T+U$$

and since I took $m^2>0$, I have that the potential has only one absolute minimum. Then from what I learned (I don't know very much yet, so this argument could be flawed) in the course of particle physics about spontaneous symmetry breaking and the correspondence between minima of the potential functional and the quantum vacuum (ground) state, I have that quantum mechanically this hamiltonian has only one, non degenerate, ground state. The picture I have in mind is the one where one graphs $(\phi_1, \phi_2, \mathcal{U}(\phi_1, \phi_2))$ where $\mathcal{U}$ is the potential energy *density*. The density in this case is a paraboloid with only one minimum because as I said one takes $m^2 > 0$ (the sign of λ doesn't matter: whatever it is the paraboloid has always the concavity facing down).

Then there exists $|\Omega\rangle$ which, up to a redefinition of the ground state energy (as one does in the free case), is an eigenstate of the Hamiltonian with zero eigenvalue:

$$H\,|\Omega\rangle = 0$$

and all other eigenvalues are greater than 0. Once I did this, I tried to find the one particle states, and to see which was the physical mass of the particles of the theory. To see this, I considered the states:

$$\begin{aligned} &a_-(\boldsymbol{p})\,|\Omega\rangle \\ &a_+(\boldsymbol{p})\,|\Omega\rangle \\ &b_-(\boldsymbol{p})\,|\Omega\rangle \\ &b_+(\boldsymbol{p})\,|\Omega\rangle \end{aligned}$$

where $a_-(\boldsymbol{p})$ is a shorthand for $a_-(t_0, \boldsymbol{p})$, and the same holds for the other three. Using the commutation relations I wrote above, one has that:

$$\begin{aligned} Ha_-(\boldsymbol{p})\,|\Omega\rangle &= -\omega_a(\boldsymbol{p})a_-(\boldsymbol{p})\,|\Omega\rangle \\ Ha_+(\boldsymbol{p})\,|\Omega\rangle &= \omega_a(\boldsymbol{p})a_+(\boldsymbol{p})\,|\Omega\rangle \\ Hb_-(\boldsymbol{p})\,|\Omega\rangle &= -\omega_b(\boldsymbol{p})b_-(\boldsymbol{p})\,|\Omega\rangle \\ Hb_+(\boldsymbol{p})\,|\Omega\rangle &= \omega_b(\boldsymbol{p})b_+(\boldsymbol{p})\,|\Omega\rangle \end{aligned}$$

then I concluded that $a_-(\boldsymbol{p})$ and $b_-(\boldsymbol{p})$ must annihilate the vacuum for every value of $\boldsymbol{p}$, otherwise the states $a_-(\boldsymbol{p})\,|\Omega\rangle$ and $b_-(\boldsymbol{p})\,|\Omega\rangle$ would have negative energy, and this would contradict what I said above about the ground state.

Then I observed that the states $a_+(\boldsymbol{p})\,|\Omega\rangle$ and $b_+(\boldsymbol{p})\,|\Omega\rangle$ have positive energy, and the eigenvalue is in accord with Einstein's energy-momentum relation[2] with mass $\sqrt{m^2 + \lambda}$ and $\sqrt{m^2 - \lambda}$, respectively. So I concluded that those are one particle states, and this field theory describes two particles with masses different from m.

4 Conclusions and open questions

Apart from the problem I spoke about at the beginning of the section 2 about the commutation relations[3], there's another problem that I consider more physical. If

[2]Sure, one should show that they have definite momentum $\boldsymbol{p}$, that is they must be eigenstates of the momentum operator P^i with eigenstate p^i, but I'll take that to be the case here. However a proof I can give is that since $[P^\mu, P^\nu] = 0$, energy eigenstates are momentum eigenstates, and from the properties of the Poincaré group I conclude that since Einstein relation holds, those states have momentum $\boldsymbol{p}$.

[3]I stress again the point that this problem is present also in the free case, that is if $\lambda = 0$. Then I don't think it is a real problem.

my reasoning about the one particle states (which used only the equations of motion and the hypothesis of the existence of a vacuum) is correct, I don't understand why there is asymmetry (one mass has a plus sign and the other a minus sign) in the particle masses, given that the equations of motion treat in the same way the two fields.

Another problem I get is that, even without knowing the commutation relations of the a and b operators, I could apply the same reasoning (using the commutations between a and b operators and the Hamiltonian) I made before to conclude that the state:

$$a_+(\boldsymbol{p}_1)\dots a_+(\boldsymbol{p}_N)\,|\Omega\rangle$$

has energy:

$$\sum_{i=1}^{N}\omega_a(\boldsymbol{p}_i)$$

and so on for the other states, just as the free case. The commutation relations of the a and b operators should be only useful to get the statistics of my particles (that I guess are bosons). With that said, I can add the fact that the observations I made before make me draw the conclusion that my Hilbert space has a Fock structure, as in the free case. Then if I use the formula that Peskin and Schroeder give for the S-matrix (equation 4.70 of their book), using the physical states (which I have a closed expression for), I obtain that S-matrix elements are 0 if the initial number of particles is different from the final number of particles, and proportional to sum of Dirac delta of physical momenta if the initial number of particles is equal from the final number of particles, multiplied by a phase which comes from applying e^{-i2Ht} to the ket on the right, which is an eigenstate. I've been a bit (a lot) sloppy here but the conclusion I make is that there is no non trivial scattering, and I don't like this because after all this is an interacting theory (I guess: maybe the answer is that by a field redefinition i can bring the potential in a diagonal form, that is something like $m_+^2\phi_+^2 + m_-^2\phi_-^2$, as I do in ordinary mechanics. If it is so this theory is free).