



## 6.6 The zero-momentum frame

We have seen in the preceding section how useful an inertial frame can be in which the total momentum vanishes. Such a frame exists uniquely for every particle system. We call it the *zero-momentum frame*,  $S_{ZM}$ , and it corresponds to the classical center-of-mass frame (cf. Exercise 6.5).

Consider an arbitrary inertial frame  $S$ , and in it a system of occasionally colliding particles subject to no forces other than very short-range forces *during* collisions (cf. footnote 2) and thus moving uniformly *between* collisions. We define the total mass  $\bar{m}$ , total momentum  $\bar{\mathbf{p}}$ , and total 4-momentum  $\bar{\mathbf{P}}$  of the system in  $S$  as the *instantaneous* sums of the respective quantities belonging to the individual particles:

$$\bar{m} = \sum m, \quad \bar{\mathbf{p}} = \sum \mathbf{p}, \quad \bar{\mathbf{P}} = \sum \mathbf{P} = \sum (\mathbf{p}, mc) = (\bar{\mathbf{p}}, \bar{m}c) \quad (6.18)$$

[cf. (6.12).] Because of the conservation laws, each of the barred quantities remains constant in time.

The quantity  $\bar{\mathbf{P}}$ , being a sum of 4-vectors, seems assured of 4-vector status itself. But, in fact, it is not quite as simple as that. If all observers agreed on which  $\mathbf{P}$ s make up the sum  $\sum \mathbf{P}$ , then  $\sum \mathbf{P}$  would clearly be a vector. But in each frame the sum is taken at one instant, which may result in different  $\mathbf{P}$ s making up the  $\sum \mathbf{P}$  of different observers. A spacetime diagram, even an imagined one, such as Fig. 5.1, is useful in proving that  $\sum \mathbf{P}$  is nevertheless a vector. A simultaneity in  $S$  corresponds to a 'horizontal' plane  $\pi$  in the diagram and a simultaneity in a second frame  $S'$  corresponds to a 'tilted' plane  $\pi'$ . In  $S$ ,  $\sum \mathbf{P}$  is summed over planes like  $\pi$ , and in  $S'$  over planes like  $\pi'$ . However, we now assert that in  $S'$  the same  $\sum \mathbf{P}$  results whether summed over  $\pi'$  or  $\pi$ . For imagine a continuous motion of  $\pi'$  into  $\pi$ . As  $\pi'$  is tilted, each individual  $\mathbf{P}$ , located at a particle on  $\pi'$ , remains constant (since the particles move uniformly between collisions) except when  $\pi'$  sweeps over a collision; but then the sub-sum of  $\sum \mathbf{P}$  which enters the collision remains constant, by 4-momentum conservation. Thus, without affecting the value, all observers *could* sum their  $\mathbf{P}$ s over the same plane  $\pi$ , and thus  $\bar{\mathbf{P}}$  is indeed a 4-vector.

Now, even if we allow some or all of the particles of the system to have zero rest-mass and thus null 4-momentum (as long as not *all* of them do *and* move in parallel), the sum  $\bar{\mathbf{P}}$  will be timelike and future-pointing—by the italicized 'corollary' of Section 5.6. As we saw in that same section, we can therefore find a frame in which the spatial components of  $\bar{\mathbf{P}}$  vanish; that is, in which  $\bar{\mathbf{p}} = 0$ . That frame is evidently the required  $S_{ZM}$ . In  $S_{ZM}$  the 4-velocity  $\mathbf{U}_{ZM}$  of  $S_{ZM}$  is  $(0, 0, 0, c)$ , so that, from (6.18),

$$\bar{\mathbf{P}} = (0, 0, 0, \bar{m}_{ZM}c) = \bar{m}_{ZM}\mathbf{U}_{ZM}, \quad (6.19)$$

where

$$\bar{m}_{ZM} = \sum m \quad \text{in } S_{ZM}, \quad (6.20)$$

obviously an invariant.

The extremities of (6.19) constitute an important 4-vector relation. Comparing it with (6.1), we see that  $\bar{m}_{ZM}$  and  $\mathbf{U}_{ZM}$  are for the system what  $m_0$  and  $\mathbf{U}$  are for a

Copyrighted material

## 118 Relativistic particle mechanics

single particle. They are the quantities that would be recognized as the rest-mass and 4-velocity of the system if its composite nature were *not* recognized (as in the case of an 'ordinary' particle—which is made up of possibly moving molecules).

Let us write out eqn (6.19) in component form in the general frame  $S$ , relative to which  $S_{ZM}$  has 3-velocity  $\mathbf{u}_{ZM}$ , say:

$$\bar{\mathbf{P}} = (\bar{\mathbf{p}}, \bar{m}c) = \bar{m}_{ZM}\gamma(u_{ZM})(\mathbf{u}_{ZM}, c).$$

From this, we can read off the following useful relations:

$$\bar{m} = \gamma(u_{ZM})\bar{m}_{ZM} \quad (6.21)$$

and

$$\bar{\mathbf{p}} = \bar{m}\mathbf{u}_{ZM}, \quad \text{or} \quad \mathbf{u}_{ZM} = \bar{\mathbf{p}}/\bar{m}. \quad (6.22)$$