1 Setting up the Mathematical Problem

1.1 Answer to Question 0

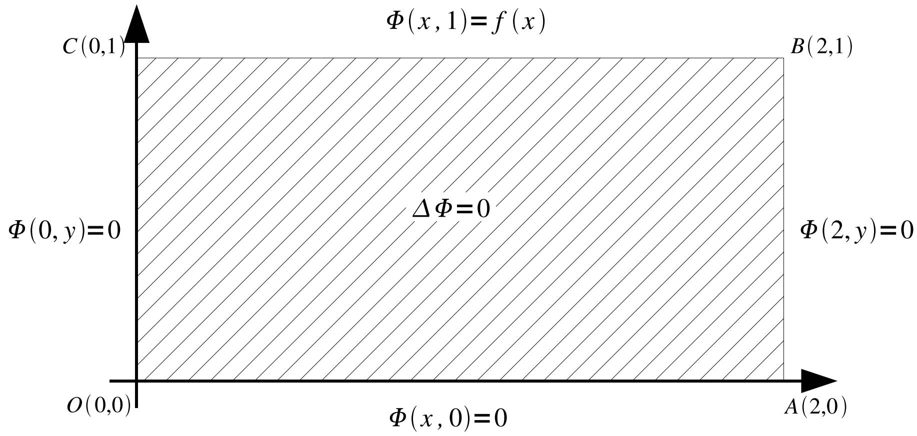


Figure 1.1:

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=0 \tag{1.1}$$

$$\begin{array}{llll} \Phi(0,y)=0 & \text{and} & \Phi(2,y)=0 & \text{for } 0\le y\le 1 \\ \Phi(x,0)=0 & \text{and} & \Phi(x,1)=f(x) & \text{for } 0\le x\le 2 \end{array}$$

2 Obtaining the Exact Solution

2.1 Answer for Question 1(i)

```
> f(x):=piecewise(x>0 and x<=3/4, 200*x/3, x>3/4 and x<=5/4, 50,
x>5/4 and x<=3/2, 100*(3-2*x), x>3/2 and x<2, 0);
```

$$f(x) = \begin{cases} \frac{200}{3}x & 0 < x \le \frac{3}{4} \\ 50 & \frac{3}{4} < x \le \frac{5}{4} \\ 100(3-2x) & \frac{5}{4} < x \le \frac{3}{2} \\ 0 & \frac{3}{2} < x < 2 \end{cases} \tag{2.1}$$

```
> f[odd](x):=piecewise(x>-2 and x<0, -subs(x=-x,f(x)), x>0 and x<2,
f(x));
```

$$f_{odd}(x) = \begin{cases} 0 & -2 < x \le -\frac{3}{2} \\ -100(3-2x) & -\frac{3}{2} < x \le -\frac{5}{4} \\ -50 & -\frac{5}{4} < x \le -\frac{3}{4} \\ -\frac{200}{3}x & -\frac{3}{4} < x \le 0 \\ \frac{200}{3}x & 0 < x \le \frac{3}{4} \\ 50 & \frac{3}{4} < x \le \frac{5}{4} \\ 100(3-2x) & \frac{5}{4} < x \le \frac{3}{2} \\ 0 & \frac{3}{2} < x < 2 \end{cases} \tag{2.2}$$

```
> plot(f[odd](x), x=0..2);
```

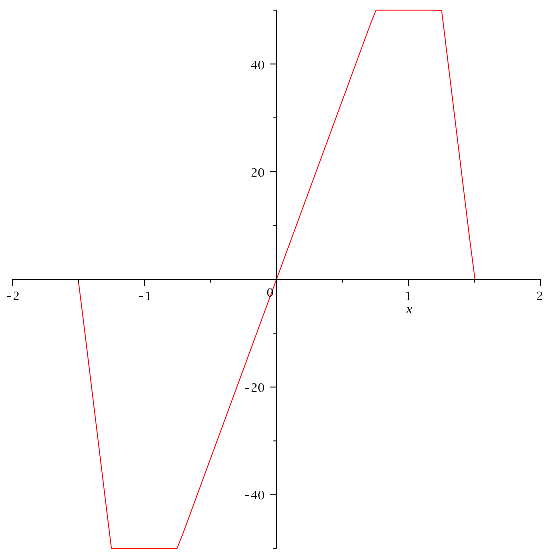


Figure 2.1: $f_{odd}(x)$

2.2 Answer for Question 1(ii)

```
> alpha[n]:=simplify(csch(n*Pi*/2)*int(f(x)*sin(n*Pi*x/2),x=0..2));
```

$$\alpha_n = \frac{800}{3} \frac{\sin\left(\frac{3n\pi}{8}\right) + 3\sin\left(\frac{5n\pi}{8}\right) - 3\sin\left(\frac{3n\pi}{4}\right)}{\sinh\left(\frac{n\pi}{2}\right) n^2 \pi^2} \tag{2.3}$$

2.3 Answer for Question 1(iii)

```
> alpha[1]:=evalf(subs(n=1,alpha[n]));
```

$$\alpha_1 = 18.48$$

```
> alpha[2]:=evalf(subs(n=2,alpha[n]));
```

$$\alpha_2 = 0.9275$$

```
> alpha[3]:=evalf(subs(n=3,alpha[n]));
```

$$\alpha_3 = -0.1970$$

```
> alpha[4]:=evalf(subs(n=4,alpha[n]));
```

$$\alpha_4 = 0.01261$$

```
> alpha[5]:=evalf(subs(n=5,alpha[n]));
```

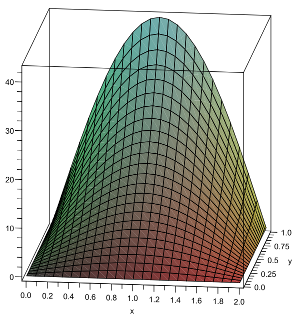
$$\alpha_5 = 0.0004956$$

2.4 Answer for Question 1(iv)

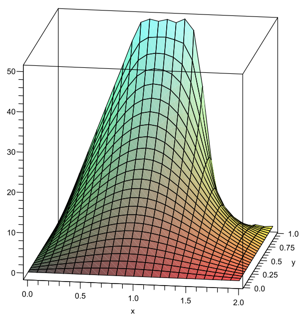
```
> Phi[N](x,y):=simplify(sum(alpha[n]*sin(n*Pi*x/2)*sinh(n*Pi*y/2),
n=1..N);
```

$$\Phi_N(x,y) = \frac{800}{3}\,\frac{\displaystyle\sum_{n=1}^{N} \frac{\left(\sin\left(\frac{3n\pi}{8}\right) + \sin\left(\frac{5n\pi}{8}\right) - \sin\left(\frac{3n\pi}{4}\right)\right)\sin\left(\frac{n\pi x}{2}\right)\sinh\left(\frac{n\pi y}{2}\right)}{\sinh\left(\frac{n\pi}{2}\right)n^2}}{\pi^2} \tag{2.4}$$

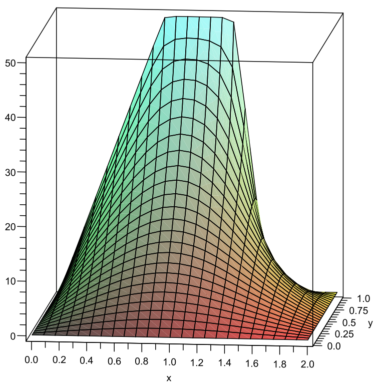
```
> plot3d(subs(N=1, Phi[N](x,y)), x=0..2, y=0..1));
> plot3d(subs(N=20, Phi[N](x,y)), x=0..2, y=0..1));
> plot3d(subs(N=50, Phi[N](x,y)), x=0..2, y=0..1));
```



(a) $\Phi_1(x,y)$



(b) $\Phi_{20}(x,y)$



(c) $\Phi_{50}(x,y)$

Figure 2.2: $\Phi_N(x,y)$ for $N = 1, 20, 50$

Figure 2.2 shows $\Phi_N(x,y)$ for $N = 1, 20, 50$. This shows that as N is increased, the accuracy of approximation to the origional function is increased. However, making N very large would require alot of time for computation, therefore N was chosen to be 50 as figure 2.2(c) shows a good approximation to $f(x)$, but is small enough to allow for easy computation.

2.5 Answer for Question 1(v)

```
> contourplot(subs(N=50, Phi[N](x,y)), x=0..2, y=0..1, coloring=[blue,red],
contours=51);
```

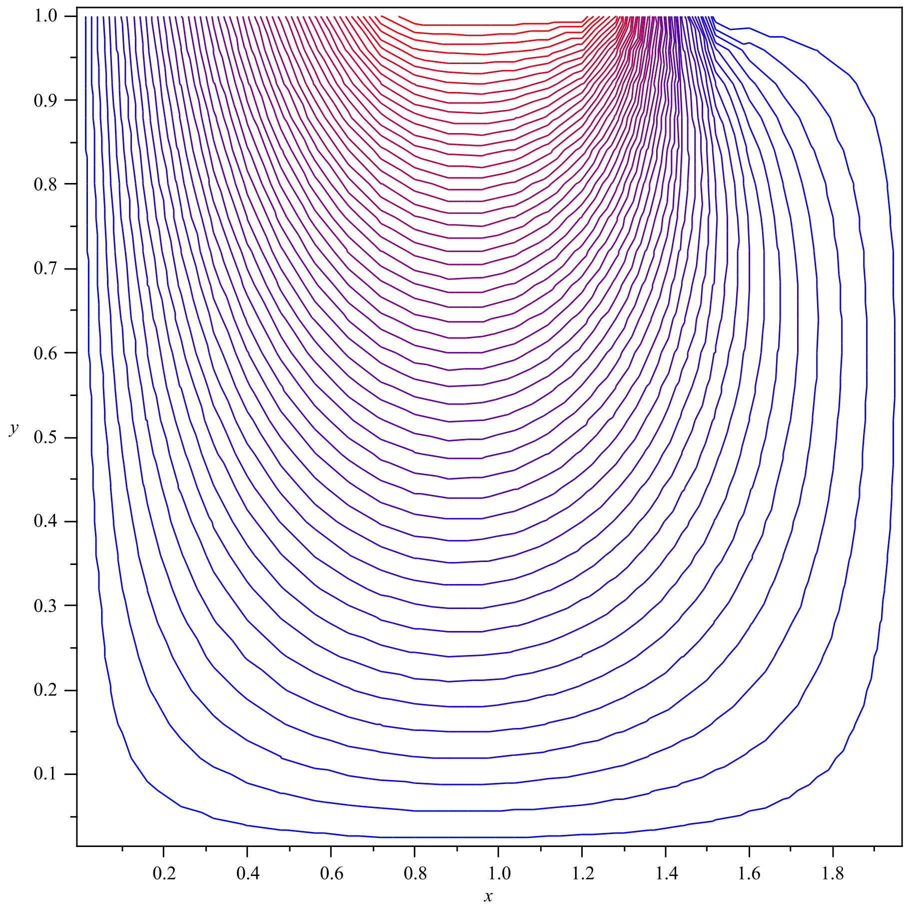


Figure 2.3:

2.6 Answer for Question 1(vi)

```
> T[50](x,y):=simplify(subs(N=50, Phi[N](x,y)));
```

$$T_{50}(x,y) = \frac{800}{3}\frac{\sum_{n=1}^{50}\frac{\left(\sin\left(\frac{3n\pi}{8}\right)+\sin\left(\frac{5n\pi}{8}\right)-\sin\left(\frac{3n\pi}{4}\right)\right)\sin\left(\frac{n\pi x}{2}\right)\sinh\left(\frac{n\pi y}{2}\right)}{\sinh\left(\frac{n\pi}{2}\right)n^2}}{\pi^2} \tag{2.5}$$

```
> T[50](1/2,1/2):=evalf(subs(x=1/2, y=1/2, T[50](x,y)));
```

$$T_{50}\left(\frac{1}{2},\frac{1}{2}\right) = 12.770$$

```
> T[50](1,1/4):=evalf(subs(x=1, y=1/4, T[50](x,y)));
```

$$T_{50}\left(1,\frac{1}{4}\right) = 7.736$$

```
> T[50](1,1/2):=evalf(subs(x=1, y=1/2, T[50](x,y)));
```

$$T_{50}\left(1,\frac{1}{2}\right) = 17.083$$

```
> T[50](1,3/4):=evalf(subs(x=1, y=3/4, T[50](x,y)));
```

$$T_{50}\left(1,\frac{3}{4}\right) = 30.379$$

```
> T[50](1,1):=evalf(subs(x=1, y=1, T[50](x,y)));
```

$$T_{50}\left(1,1\right) = 50.013$$

```
> T[50](3/2,1/2):=evalf(subs(x=3/2, y=1/2, T[50](x,y)));
```

$$T_{50}\left(\frac{3}{2},\frac{1}{2}\right) = 8.440$$

2.7 Answer for Question 1(vii)

The centre of the plate is located at $\left(1, \frac{1}{2}\right)$. The temperature at this point has already been calculated in section 2.6 and is shown in equation 2.6.

$$k = 17.083 \tag{2.6}$$

Using Maple to solve $f(x) = k$ for x gives:

```
> solve(f(x)=k, x);
```

$$x = 0.2562,\ 1.4146$$

Since both of these occur on the top boundary (i.e. $y = 1$), the coordinates at which the boundary of the plate is equal to $k°C$ are (0.2562, 1) and (1.4146, 1).

2.8 Answer for Question 1(viii)

```
> q[N](x):=simplify(subs(y=x/2, Phi[N](x,y)));
```

$$q_N(x) = -\frac{800}{3}\frac{\displaystyle\sum_{n=1}^{N}\frac{\left(-\sin\left(\frac{3n\pi}{8}\right)-\sin\left(\frac{5n\pi}{8}\right)+\sin\left(\frac{3n\pi}{4}\right)\right)\sin\left(\frac{n\pi x}{2}\right)\sinh\left(\frac{n\pi x}{4}\right)}{\sinh\left(\frac{n\pi}{2}\right)n^2}}{\pi^2} \tag{2.7}$$

```
> plot(subs(N=50, q[N](x,y)), x=0..2)
```

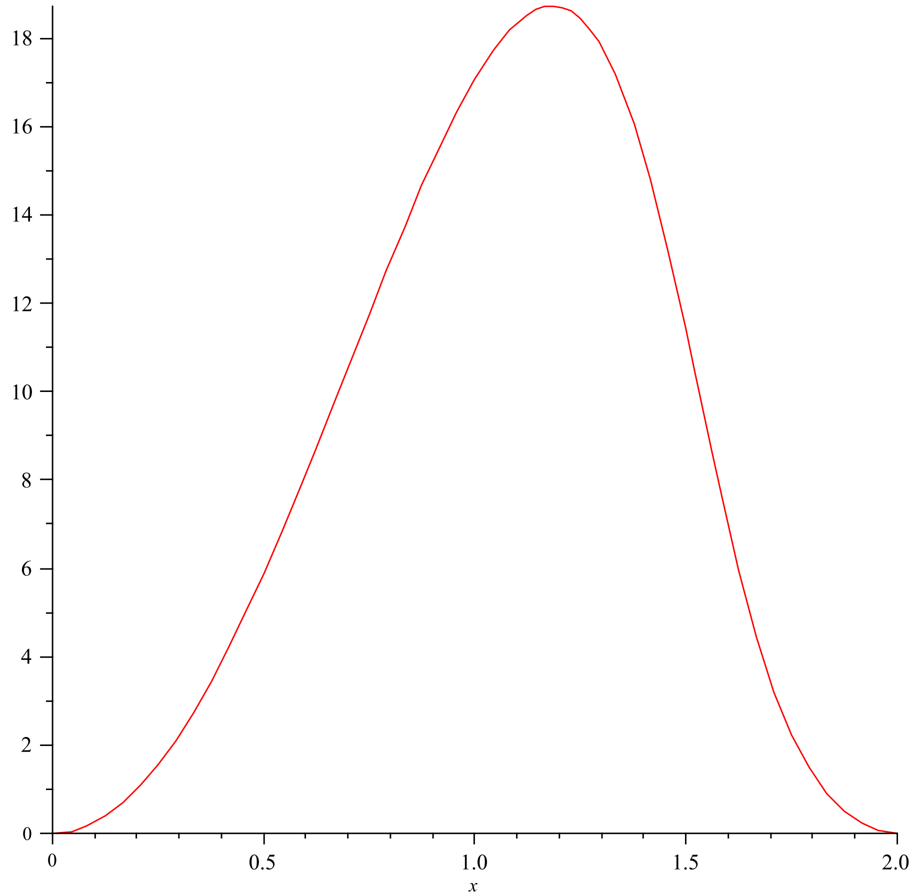


Figure 2.4: $q_{50}(x)$

3 Obtaining a Numerical Solution

3.1 Answer for Question 2(i)

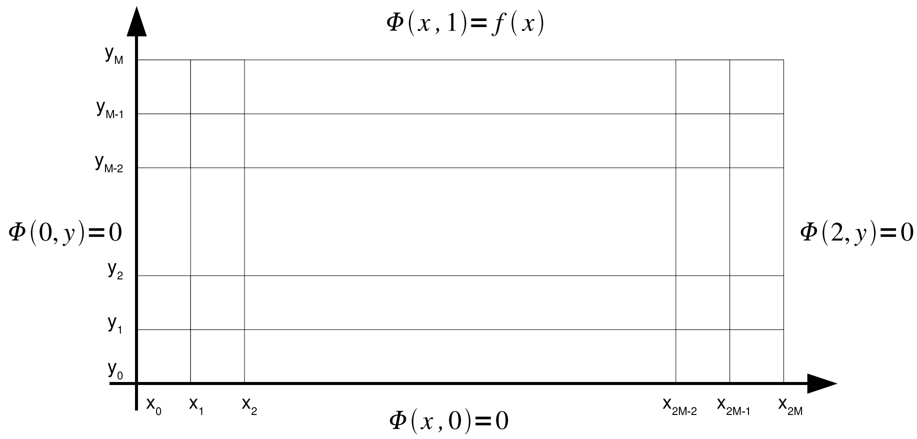


Figure 3.1:

The number of unknown variables in this problem is represented by the number of internal grid points. Expressing this as a function of M (the number of intervals along the y-axis) gives equation 3.1.

$$(2M-1)(M-1)$$
$$2M^2 - 3M + 1 \tag{3.1}$$

3.2 Answer for Question 2(ii)

The standard finite difference approximation to Laplace's equation for $\Phi(x, y)$ is given in equation 3.2.

$$\Phi_{i+1,j} + \Phi_{i-1,j} + \Phi_{i,j+1} + \Phi_{i,j-1} - 4\Phi_{i,j} = 0 \tag{3.2}$$

3.2.1 Answer for Question 2(ii)(a)

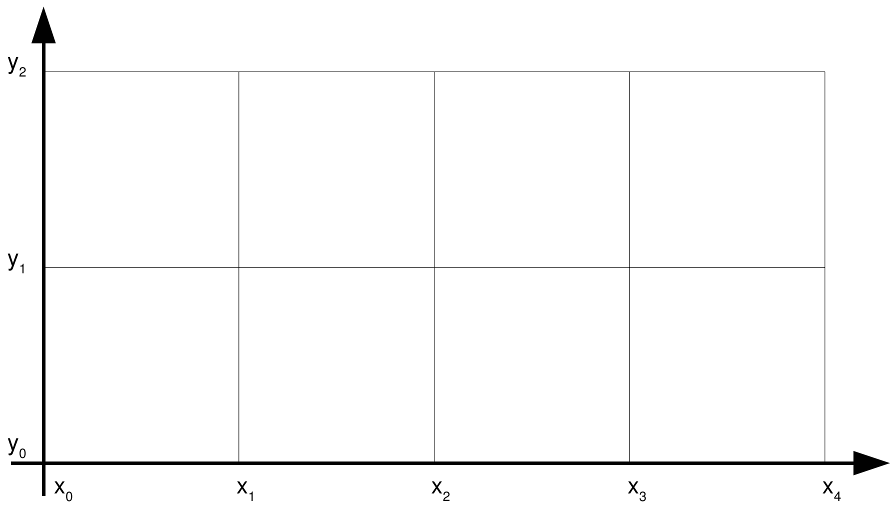


Figure 3.2:

For $j = 1$ and $i = 1$:

$$\Phi_{2,1} + \Phi_{0,1} + \Phi_{1,2} + \Phi_{1,0} - 4\Phi_{1,1} = 0$$

$$-4\Phi_{1,1} + \Phi_{2,1} = -33.3 \tag{3.3}$$

For $j = 1$ and $i = 2$:

$$\Phi_{3,1} + \Phi_{1,1} + \Phi_{2,2} + \Phi_{2,0} - 4\Phi_{2,1} = 0$$

$$\Phi_{1,1} - 4\Phi_{2,1} + \Phi_{3,1} = -50 \tag{3.4}$$

For $j = 1$ and $i = 3$:

$$\Phi_{4,1} + \Phi_{2,1} + \Phi_{3,2} + \Phi_{3,0} - 4\Phi_{3,1} = 0$$

$$\Phi_{2,1} - 4\Phi_{3,1} = 0 \tag{3.5}$$

Matrix equation:

$$\begin{bmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} \Phi_{1,1} \\ \Phi_{2,1} \\ \Phi_{3,1} \end{bmatrix} = \begin{bmatrix} -33.3 \\ -50 \\ 0 \end{bmatrix} \tag{3.6}$$

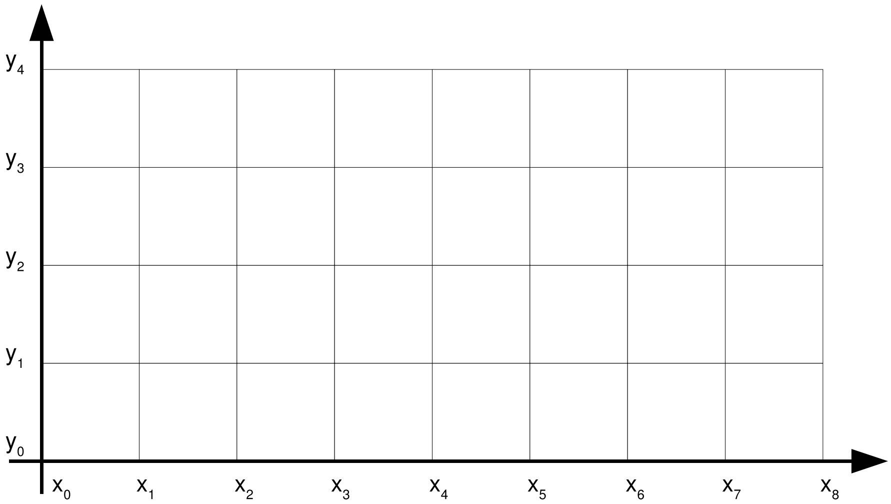


Figure 3.3:

3.2.2 Answer for Question 2(ii)(b)

For $j = 3$ and $i = 1$:

$$\Phi_{2,3} + \Phi_{0,3} + \Phi_{1,4} + \Phi_{1,2} - 4\Phi_{1,3} = 0$$

$$\Phi_{1,2} - 4\Phi_{1,3} + \Phi_{2,3} = -16.7 \tag{3.7}$$

For $j = 3$ and $i = 2$:

$$\Phi_{3,3} + \Phi_{1,3} + \Phi_{2,4} + \Phi_{2,2} - 4\Phi_{2,3} = 0$$

$$\Phi_{2,2} + \Phi_{1,3} - 4\Phi_{2,3}\Phi_{3,3} = -33.3 \tag{3.8}$$

For $j = 3$ and $i = 3$:

$$\Phi_{4,3} + \Phi_{2,3} + \Phi_{3,4} + \Phi_{3,2} - 4\Phi_{3,3} = 0$$

$$\Phi_{3,2} + \Phi_{2,3} - 4\Phi_{3,3} + \Phi_{4,3} = -50 \tag{3.9}$$

For $j = 3$ and $i = 4$:

$$\Phi_{5,3} + \Phi_{3,3} + \Phi_{4,4} + \Phi_{4,2} - 4\Phi_{4,3} = 0$$

$$\Phi_{4,2} + \Phi_{3,3} - 4\Phi_{4,3}\Phi_{5,3} = -50 \tag{3.10}$$

For $j = 3$ and $i = 5$:

$$\Phi_{6,3} + \Phi_{4,3} + \Phi_{5,4} + \Phi_{5,2} - 4\Phi_{5,3} = 0$$

$$\Phi_{5,2} + \Phi_{4,3} - 4\Phi_{5,3} + \Phi_{6,3} = -50 \tag{3.11}$$

For $j = 3$ and $i = 6$:

$$\Phi_{7,3} + \Phi_{5,3} + \Phi_{6,4} + \Phi_{6,2} - 4\Phi_{6,3} = 0$$

$$\Phi_{6,2} + \Phi_{5,3} - 4\Phi_{6,3} + \Phi_{7,3} = 0 \tag{3.12}$$

For $j = 3$ and $i = 7$:

$$\Phi_{8,3} + \Phi_{6,3} + \Phi_{7,4} + \Phi_{7,2} - 4\Phi_{7,3} = 0$$

$$\Phi_{7,2} + \Phi_{6,3} - 4\Phi_{7,3} + \Phi_{8,3} = 0 \tag{3.13}$$

Matrix equation:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} \Phi_{1,2} \\ \Phi_{2,2} \\ \Phi_{3,2} \\ \Phi_{4,2} \\ \Phi_{5,2} \\ \Phi_{6,2} \\ \Phi_{7,2} \\ \Phi_{1,3} \\ \Phi_{2,3} \\ \Phi_{3,3} \\ \Phi_{4,3} \\ \Phi_{5,3} \\ \Phi_{6,3} \\ \Phi_{7,3} \end{bmatrix} = \begin{bmatrix} -16.7 \\ -33.3 \\ -50 \\ -50 \\ -50 \\ 0 \\ 0 \end{bmatrix} \tag{3.14}$$

3.3 Answer for Question 2(iii)

3.3.1 Answer for Question 2(iii)(a)

Equation 3.15 shows the initial estimate of temperature at $\Phi_{1,1}$.

$$\Phi_{1,1} = \frac{\Phi_{1,2} - \Phi_{1,0}}{2}$$
$$\Phi_{1,1} = 16.7 \tag{3.15}$$

Equation 3.16 shows the initial estimate of temperature at $\Phi_{2,1}$.

$$\Phi_{2,1} = \frac{\Phi_{2,2} - \Phi_{2,0}}{2}$$
$$\Phi_{2,1} = 25 \tag{3.16}$$

Equation 3.17 shows the initial estimate of temperature at $\Phi_{3,1}$.

$$\Phi_{3,1} = \frac{\Phi_{3,2} - \Phi_{3,0}}{2}$$
$$\Phi_{3,1} = 0 \tag{3.17}$$

3.3.2 Answer for Question 2(iii)(b)

The Gauss-Seidel iteration equations are:

$$\Phi_{1,1}^{k+1} = \frac{\Phi_{2,1}^{k} + 33.3}{4} \tag{3.18}$$

$$\Phi_{2,1}^{k+1} = \frac{\Phi_{1,1}^{k+1} + \Phi_{3,1}^{k} + 50}{4} \tag{3.19}$$

$$\Phi_{3,1}^{k+1} = \frac{\Phi_{2,1}^{k+1}}{4} \tag{3.20}$$

Using these equations convergence correct to 3 decimal places was achieved after 7 iterations as shown in table 1.

k	$\Phi_{1,1}^{k}$	$\Phi_{2,1}^{k}$	$\Phi_{3,1}^{k}$
0	16.6667	25.0000	0.0000
1	14.5833	16.1458	4.0365
2	12.3698	16.6016	4.1504
3	12.4837	16.6585	4.1646
4	12.4980	16.6656	4.1664
5	12.4997	16.6665	4.1666
6	12.5000	16.6667	4.1667
7	12.5000	16.6667	4.1667
8	12.5000	16.6667	4.1667

Table 1: Gauss-Seidel iterations

The temperatures at points $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\left(1, \frac{1}{2}\right)$ and $\left(\frac{3}{2}, \frac{1}{2}\right)$ are given by $\Phi_{1,1}$, $\Phi_{2,1}$

and $\Phi_{3,1}$ respectively. The temperatures at points $(1, 1)$, $\left(\frac{3}{2}, 0\right)$ are given by the boundary conditions of the plate. Therefore the temperatures are as follows:

$$\Phi\left(\frac{1}{2}, \frac{1}{2}\right) = 12.5$$
$$\Phi\left(1, \frac{1}{2}\right) = 16.6667$$
$$\Phi\left(1, 1\right) = 50$$
$$\Phi\left(\frac{3}{2}, \frac{1}{2}\right) = 4.1667$$
$$\Phi\left(\frac{3}{2}, 0\right) = 0$$

4 Comparison of the Exact Solution and the Numerical Solution

4.1 Answer for Question 3