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Numerical Study of Discrete Plane Area-preserving Mappings

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Summary. Dynamical systems with two degrees of freedom can be reduced to the study of a two-dimensional mapping. Here, we consider discrete mappings operating only on integers. This allows an exact numerical study, without round-off errors. Any point belongs to a finite cycle. These mappings are strictly one-to-one.
We define, for comparison, a "random mapping" and we give its principal properties. In some cases, these properties are very similar to the observed properties of definite mappings.

Key words: surface of section – plane area preserving mapping – ergodic properties – numerical experiments – dynamical systems

I. Introduction

Some problems of celestial mechanics are equivalent to the study of a dynamical system with two degrees of freedom for which the Hamiltonian does not depend on the time. A method to study this problem is the method of the "surface of section". It consists essentially in considering not a complete trajectory in the phase-space, but only its successive intersections with a certain "surface of section". All the most interesting characteristics of the trajectories of the dynamical problem are reflected into equivalent properties of the sequence of points thus obtained. This method has been introduced by Poincaré (1892) and used afterwards by Birkhoff (1917) for theoretical purposes. Then it has been used in the numerical study of many different problems (see for instance Hénon and Heiles, 1964).
Poincaré has shown that there is an area preserving mapping T between these points:

$$P_0, P_1 = T(P_0), \ldots, P_n = T^n(P_0).$$

But it is in general impossible to write this mapping as a function of the hamiltonian. We are obliged to choose a particular mapping and we study the set of points obtained by repeated applications of the mapping T.
For instance, let us consider the following mapping

$$T\begin{cases} x_{i+1} = x_i + y_i + 1 - \cos y_i & \mod 2\pi \\ y_{i+1} = y_i - \lambda(\sin x_{i+1} + 1 - \cos x_{i+1}) & \mod 2\pi \end{cases} \tag{1}$$

$$-\pi < x \leq +\pi \quad -\pi < x \leq +\pi .$$

In the last section (V) we shall give the reasons for this choice. The determinant of the jacobian matrix is equal to 1. This mapping has two invariant points $I_1(x=0, y=0)$ and $I_2(x=-\pi/2, y=0)$. λ is a positive parameter. For $\lambda > 4$, I_1 and I_2 are unstable; for $\lambda < 4$, I_1 is stable and I_2 is unstable (Rannou, 1972).
Figure 1 shows typical sets of points for some initial conditions (x_0, y_0) and for $\lambda = 1.30$. 1000 points are plotted for each of these five sequences. We can see two regions in the plane. Round the invariant point I_1, the points seem to lie on invariant curves (two sequences). The curves are deforming and diluting (one sequence). By degrees, they make an "ergodic" figure (two sequences). This figure is not symmetrical with respect to the invariant point I_1.

II. A Discrete Mapping T^*

1. Motivations

a) In a numerical study a computer is used to calculate the coordinates of the points by repeated applications of a mapping T:

$$\begin{cases} x_{i+1} = f(x_i, y_i) \\ y_{i+1} = g(x_i, y_i) \end{cases} \tag{2}$$

where f and g are real functions. We call this the *continuous case.*
But the computer retains only a fixed number of digits and makes round-off errors. Another computer will not give the same results. In the "ergodic" part of the plane after many applications of the mapping T, the

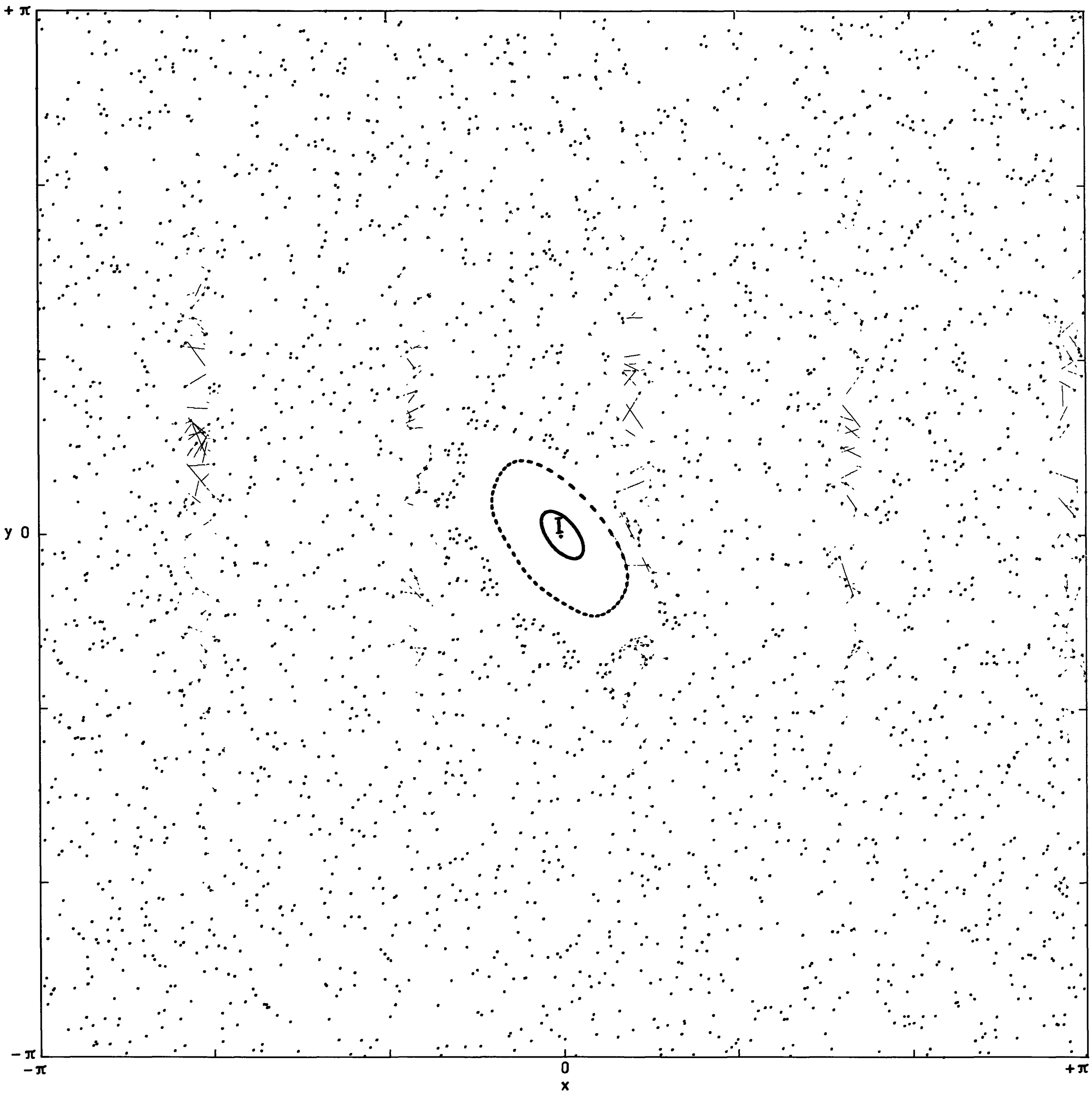

Fig. 1. Five sets of point for $\lambda = 1.30$, obtained by repeated applications of the mapping $\boldsymbol{T}$ [see Eq. (1)]

resulting error increases exponentially. We obtain approximate values only. To avoid this, we shall try to define a mapping which operates only on integers. The computer will then give exact and reproducible results. We call this the *discrete case*.

b) In fact, that is more or less what the computer does, but in an uncontrolled way. To define a mapping in the discrete case is equivalent to giving explicit rules of round-off to the computer.

c) The mathematical mapping $\boldsymbol{T}$ is a bijection. From one point (x_i, y_i), we obtain by application of $\boldsymbol{T}$ only one other point (x_{i+1}, y_{i+1}) and conversely. But the mapping studied on the computer gives sometimes the same point for two different points because of the round-off errors. It is possible to define a mapping $\boldsymbol{T}^*$ in the discrete case which is strictly one-to-one as we shall see below.

d) In the continuous case, the number of points which can be obtained by repeated applications of the mapping is generally infinite. We must choose arbitrarily the total number of points plotted for each value of the initial point. On the contrary in the discrete case, the number of accessible points is finite. Any set of points always goes back to the initial point.

e) For the same reason in the discrete case, we can explore all the accessible points, the only limitation being computer storage.
f) In the discrete case, we can define rigorously a "random mapping" and find its principal properties (see Section III).

2. An Example

We shall show how to define a discrete mapping $\boldsymbol{T}^*$ derived from the continuous mapping $\boldsymbol{T}$ given by the Eq. (1) (see Fig. 2). We divide the x axis between $-\pi$ and $+\pi$ in m equal segments, m being an even number. We write $x=\frac{2\pi}{m}a$. We do the same thing with the y axis between $-\pi$ and $+\pi$ with $y=\frac{2\pi}{m}b$. a and b are integers on the nodes of the square lattice thus defined. We call m the "*discretisation*". We substitute these x and y values in the expression of the mapping $\boldsymbol{T}$. We obtain:

$$\begin{aligned} a_{i+1} &= a_i + b_i + \frac{m}{2\pi}\left(1-\cos\frac{2\pi}{m}b_i\right) && \bmod m \\ b_{i+1} &= b_i - \frac{\lambda m}{2\pi}\left(\sin\frac{2\pi}{m}a_{i+1} + 1 - \cos\frac{2\pi}{m}a_{i+1}\right) && \bmod m \end{aligned} \tag{3}$$

a_i and b_i are integers. We want a_{i+1} and b_{i+1} to be also integers. Therefore we define our mapping $\boldsymbol{T}^*$ by:

$$\boldsymbol{T}^* \begin{cases} a_{i+1} = a_i + b_i + \left[\frac{m}{2\pi}\left(1-\cos\frac{2\pi}{m}b_i\right)\right] & \bmod m \\ b_{i+1} = b_i - \left[\frac{\lambda m}{2\pi}\left(\sin\frac{2\pi}{m}a_{i+1} + 1 - \cos\frac{2\pi}{m}a_{i+1}\right)\right] & \\ & \bmod m \end{cases} \tag{4}$$

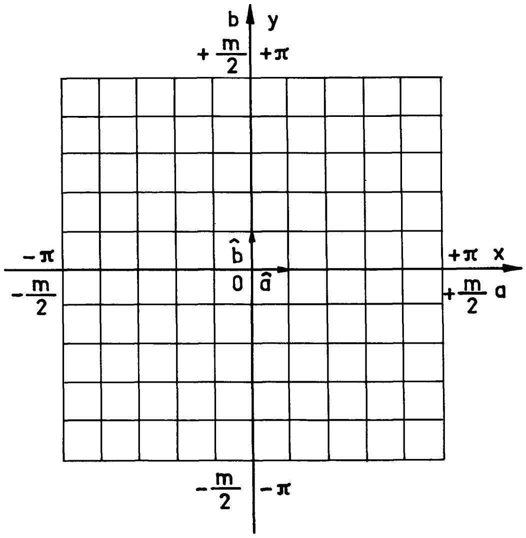

Fig. 2. Square lattice used to define a discrete mapping

where $[X]$ represents the integer nearest to X. If X is exactly equal to $c+0.5$, c being an integer, the computer chooses $[X]=c+1$.

3. Some Properties of $\boldsymbol{T}^*$

a) The mapping $\boldsymbol{T}^*$ is a one-to-one mapping of the (a, b) space over itself.
b) Because the number of accessible points is finite, a set of points P_0, $P_1=\boldsymbol{T}^*(P_0)$, ..., $P_n=\boldsymbol{T}^{*n}(P_0)$ is coming back.
c) A set of points cannot come back on another point that on the initial point P_0. One point can belong only to one cycle.
d) All the points reached by repeated applications of the mapping $\boldsymbol{T}^*$ from an initial point P_0, make a *cycle*. It is clear that each point of a cycle can be the initial point of that cycle.
e) The mapping $\boldsymbol{T}^*$ has two invariant points $I_1(0,0)$ and $I_2(-\frac{m}{4},0)$, if m is a multiple of 4. For $\lambda>4$, other invariant points can appear. These points can also be considered as cycles of length 1.
The following figures show some examples. Figure 3 shows one cycle around the invariant point I_1 for $\lambda=1.30$ and $m=400$. It has roughly the shape of an ellipse, with an appreciable thickness. Its length is 382 points.
Figure 4 shows a cycle of 100384 points for the same mapping. The points seem to fill the plane in a random way with the exception of an oval region around the invariant point I_1. That region is occupied by short cycles of the kind shown by Fig. 3. Thus there are two parts in the plane: a little region around the invariant point (0, 0) and the rest of the plane that seems to be "ergodic". For $\lambda=10$, the situation is quite different. Figure 5 shows a cycle of 104037 points. All these points seem to be plotted in a random way in the entire plane. This figure suggests that we should try to define and study a "random mapping".

III. A Random Mapping

In the discrete case, the total number of accessible points is $M=m^2$, if m is the "discretisation". A one-to-one mapping can be considered as a *permutation* of these M points. It is possible to make $M!$ permutations of M points, therefore there are $M!$ possible mappings. We define a *random mapping* $\mathscr{T}$ simply by attributing the same probability $1/M!$ to each actual mapping.
This random mapping has the following properties:
1. the probability to obtain a cycle of given length n from a given point A_0 is

$$p_n = 1/M\,.$$

Note that p_n is independent of n.
2. the average length of the cycle originating at a given point is equal to $\frac{M+1}{2}$.

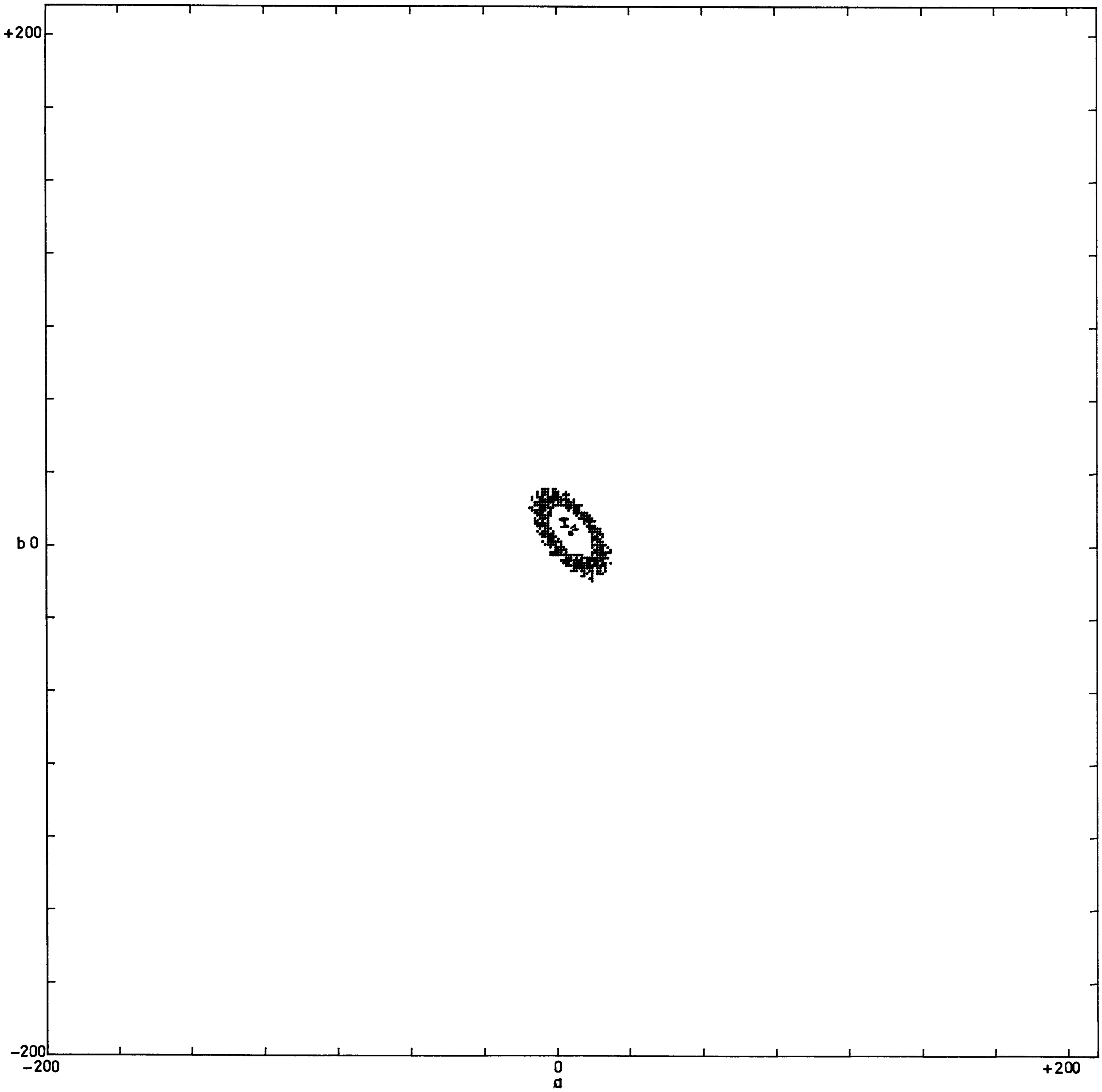

Fig. 3. A cycle of 382 points obtained with T^* for $\lambda = 1.30$ and $m = 400$. It lies in the oval region shown in Fig. 4

3. the average number of cycles of all lengths is nearly equal to $\log_e M + \gamma$, where γ is Euler's constant.
The proofs are not difficult. They will not be given here (see Rannou, 1972).

IV. Numerical Results

We have considered two cases:

for $\lambda = 1.30$ $I_1(0,0)$ is an invariant stable point and $I_2(-\frac{m}{4}, 0)$ is an invariant unstable point if m is multiple of 4.

for $\lambda = 10$ I_1 and I_2 are invariant unstable points.

1. Chart of Initial Points

Arbitrarily, we have chosen to explore the accessible points from bottom to top, and from the left to the right. The coordinates of the initial point of the first cycle are ($a = 1 - \frac{m}{2}$, $b = 1 - \frac{m}{2}$). Any point that is found in this exploration and does not belong to a cycle already obtained, is the initial point of a new cycle. These points are plotted on the Fig. 6 for $\lambda = 1.30$ and $m = 800$. We see that they fill two different regions near the invariant point I_1, and at the bottom of the plane. This clearly corresponds to the two regions already found (Fig. 3 and 4). In the case $\lambda = 10$, we find

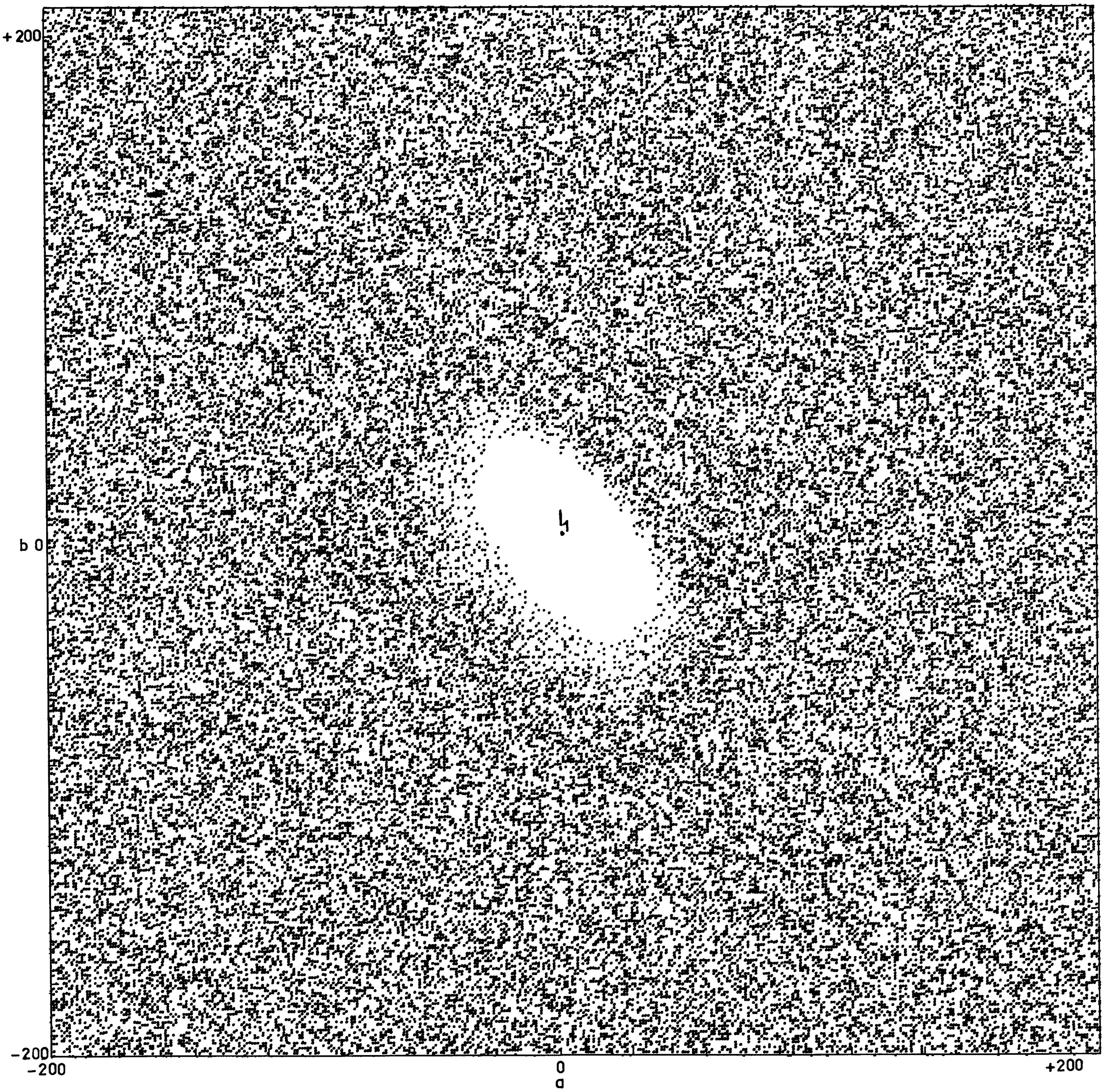

Fig. 4. A cycle of 100384 points obtained with the discrete mapping T^* [see Eq. (4)] for $\lambda = 1.30$ and $m = 400$

only an "ergodic" region and the initial points are all at the bottom of the plane.

2. *Total Number of Cycles*

$\lambda = 1.30$.

The observed total number of cycles is larger than the number calculated for a random mapping, because of the stability of the invariant point I_1. Around this point, there are short cycles. But with the help of charts of the kind shown by Fig. 6, we can easily discriminate cycles belonging to the ergodic region to the small "stable" region around I_1, and count them separately. The result is shown in the last two columns of Table 1.

In the ergodic region, the number of cycles is close to $\log_e m^2 + \gamma$.

We can neglect the non ergodic region which has a small area. In the short cycles region, the number of cycles is larger and approximately proportional to m.

$\lambda = 10$.

For $\lambda = 10$, the total number of cycles is very similar to that of a random mapping, as shown by the following table.

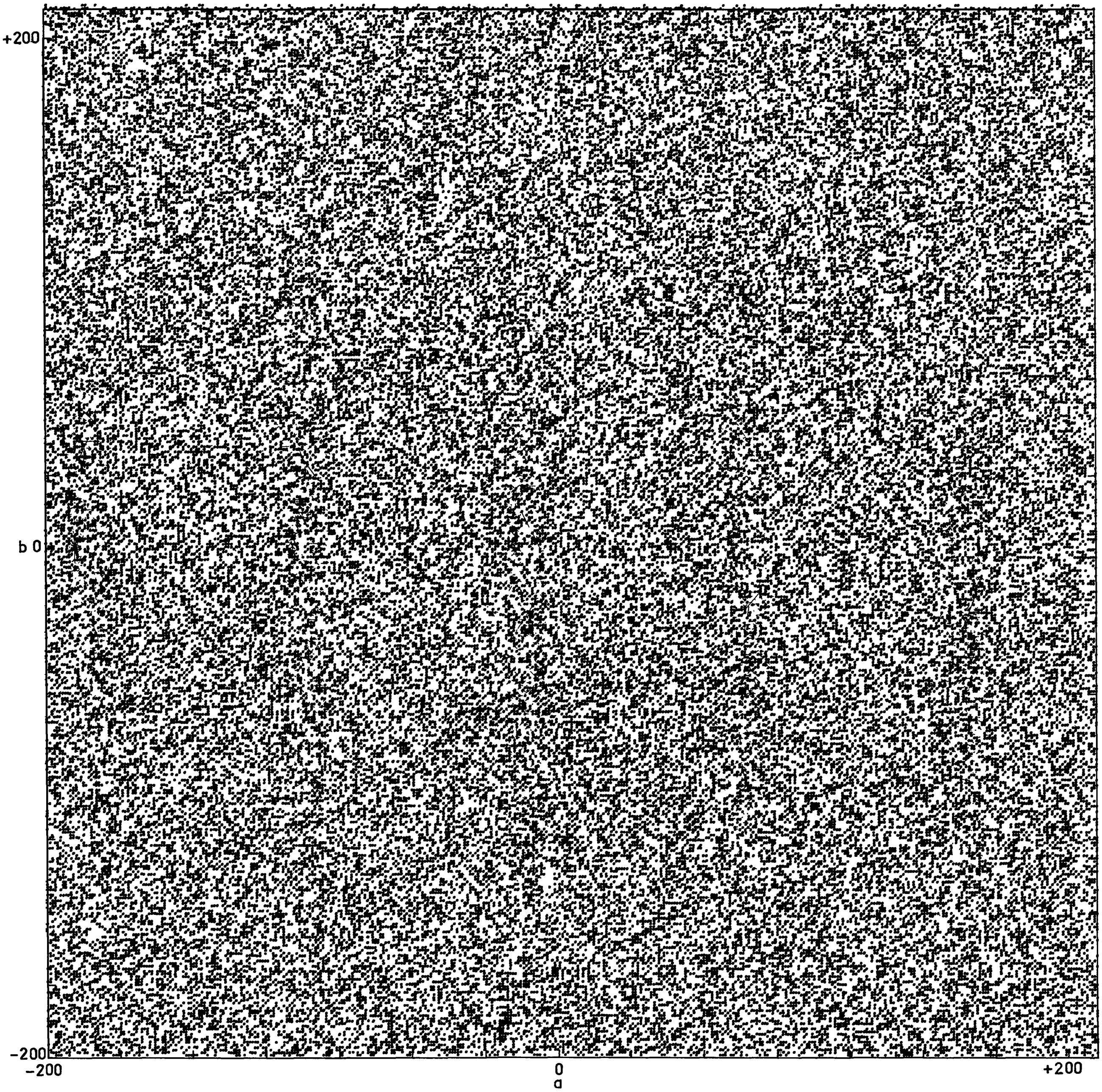

Fig. 5. A cycle of 104037 points obtained with T^* for $\lambda = 10$ and $m = 400$

Table 1.

Discretisation m	Total number of cycles: Random mapping	Numerical experiments: "Ergodic" region	Numerical experiments: Short cycles region
300	11.9	8	33
400	12.4	9	48
500	12.9	12	59
600	13.4	12	60
700	13.6	14	67
800	13.8	10	88

Total number of cycles for $\lambda = 1.30$ and different values of the discretisation m

Table 2.

m	Total number of cycles: Random mapping	Total number of cycles: Numerical experiments
300	11.98	11
400	12.56	12
500	13.01	11
600	13.37	20
700	13.68	13
800	13.95	12

Total number of cycles for $\lambda = 10.$ and different values of the discretisation m.

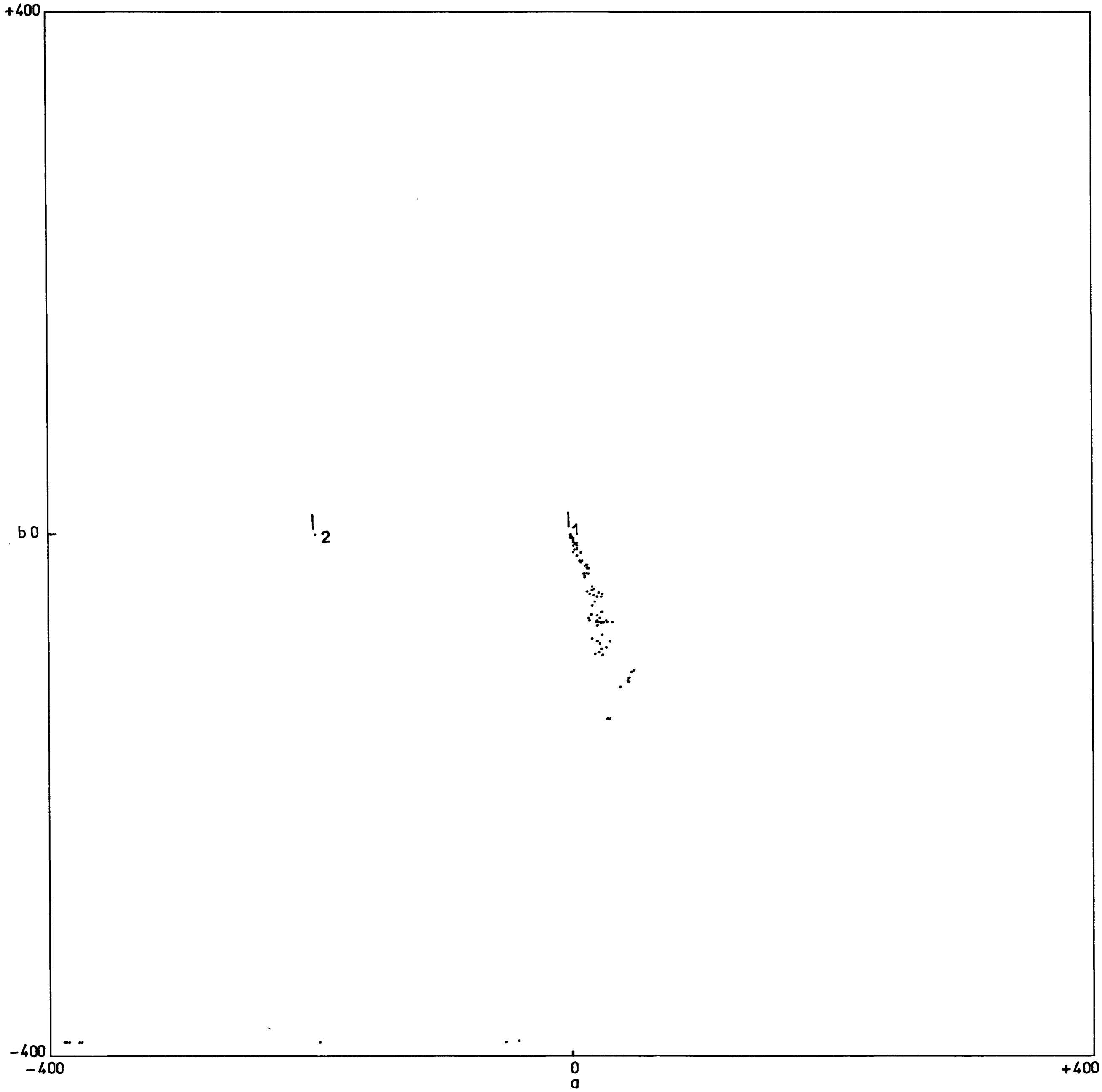

Fig. 6. Chart of the initial points for $\lambda = 1.30$ and $m = 800$

3. *Longest Cycle and Average Length*

The length of the longest cycle is between 65% and 85% of the accessible points. It lies always in the "ergodic" region. For example, λ is 10, m is 800. The longest cycle has 435570 points and represents 68.05% of the accessible points. Table 3 shows that the average length obtained by numerical experiments is similar to that calculated with a random mapping for $\lambda = 10$. It is the average length of the cycle from a given initial point i.e. $\Sigma n^2/M$ where $M = m^2$ and n represents the length of a cycle.

Table 3

Discretisation	Average length	
m	Random mapping	Numerical experiments
300	45000	49540
400	80000	77663
500	125000	187508
600	180000	190215
700	245000	162770
800	320000	355709

Average length for $\lambda = 10$. Comparison between a random mapping and numerical experiments on T^*.

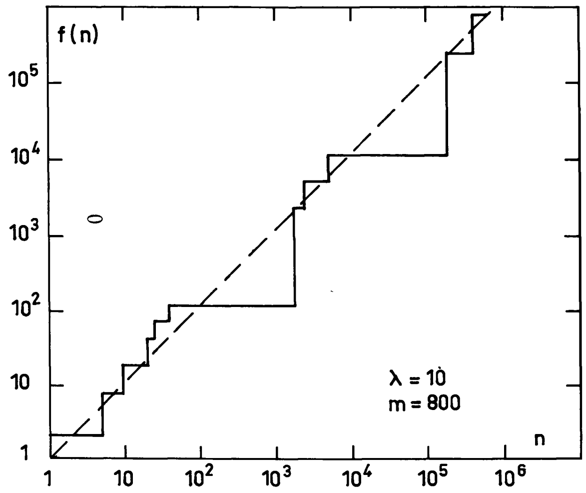

Fig. 7. Comparison between the cumulative distribution $f(n)$ for T^* (full line) and for the random mapping $\mathscr{T}$ (broken line)

For $m = 700$ and $\lambda = 10.$, the results are not very similar because there are two longest cycles that are symmetrical with respect to the invariant point I_1.

4. Cumulation Distribution of Cycles

We compute the number of points that belong to cycles of length less than, or equal to n, as a function of n, which we call $f(n)$. The greatest value of n is m^2, the total number of accessible points.

In the case of a random mapping $\mathscr{T}$, the average number of cycles containing n points is $1/n$. Thus the average number of points belonging to cycles of n points is 1, and the average number of points belonging to cycles of length less than or equal to n is $f(n) = n$ (see Rannou, 1972).

Figure 7 shows with logarithmic scales, the theoretical function $f(n)$ and the observed function. The two functions are quite similar. There λ is 10 and m is 800. Thus, we see that the properties of the mapping T^* given by Eq. (4) are very similar to those of a random mapping. The total number of cycles is small. The detailed distribution of cycle lengths is in agreement with the theory.

V. Remarks on the Choice of a Discrete Mapping

The mapping given by Eq. (4) that we have studied derives from the mapping introduced by Taylor (1969):

$$T_0 \begin{cases} x_{i+1} = x_i + y_i & \mod 2\pi \\ y_{i+1} = y_i - \lambda \sin(x_i + y_i) & \mod 2\pi \end{cases} \qquad (5)$$

$$-\pi < x \leq +\pi \quad -\pi < y \leq +\pi .$$

In fact, our work on discrete mappings started with the form (5) (Rannou, 1972). Later, however, it became clear that this mapping has hidden symmetries and degeneracies which produce particular properties, and therefore is not a typical representative of mappings at large. This will be explained in the present section. The mapping T_0 was therefore replaced by the more complicated mapping T [Eq. (1)], in which the symmetries are eliminated, and which appears to be free from degeneracy. We proceed now to the study of T_0. The determinant of the Jacobian matrix is equal to 1. This mapping has two invariant points $I_1(x=0,\ y=0)$ and $I_2(x=+\pi, y=0)$. λ is a positive parameter. For $\lambda > 4$, I_1 and I_2 are unstable; for $\lambda < 4$, I_1 is stable and I_2 is unstable.

1. Description

In Fig. 8, we have plotted nine sets of points, for $\lambda = 1.30$. We can compare this figure with Fig. 1. We see three regions in the plane. The invariant curve region round the invariant point I_1 is larger on Fig. 8, and the curves are symmetrical with respect of the point I_1. The ergodic region is smaller on Fig. 8 and between these two regions, we see cycles that are deformed and they are diluting to make by degrees, the "ergodic" region.

Now, we define the discrete mapping T_0^* in the same way as above (see Section III).

$$T_0^* \begin{cases} a_{i+1} = a_i + b_i & \mod m \\ b_{i+1} = b_i - \left[\dfrac{\lambda m}{2\pi} \sin \dfrac{2\pi}{m}(a_i + b_i)\right] & \mod m \end{cases} \qquad (6)$$

a_i and b_i are again integers.

Figure 9 shows two cycles for $\lambda = 1.30$ and $m = 800$. One cycle is made of "islands". It has 6180 points and it is the longest cycle. In the middle, there is a cycle that resembles an ellipse with knots around the invariant point I_1.

Figure 10 shows one cycle for $\lambda = 1.30$ and $m = 700$. The length of this cycle is 3878 points. This set of points fills up a large part of the plane but the density of points is much less than on Fig. 4.

Figure 11 shows again one cycle for $\lambda = 10.$ and $m = 700$. This cycle is constituted by 3218 points. These points seem to fill the plane in a quasi random way, but also here, the density of points is much less than on Fig. 5.

2. The Mapping T_0^* is Very Different from a Random Mapping

Indeed, in the two cases $\lambda = 1.30$ and $\lambda = 10$, we find too many cycles and only comparatively short cycles. The Table 4 shows the results for $\lambda = 10$.

The mapping T_0^* is very different from a random mapping. This fact was very puzzling to us for a while, until we found the explanation which is as follows. The mapping T_0 is the product of two plane area-

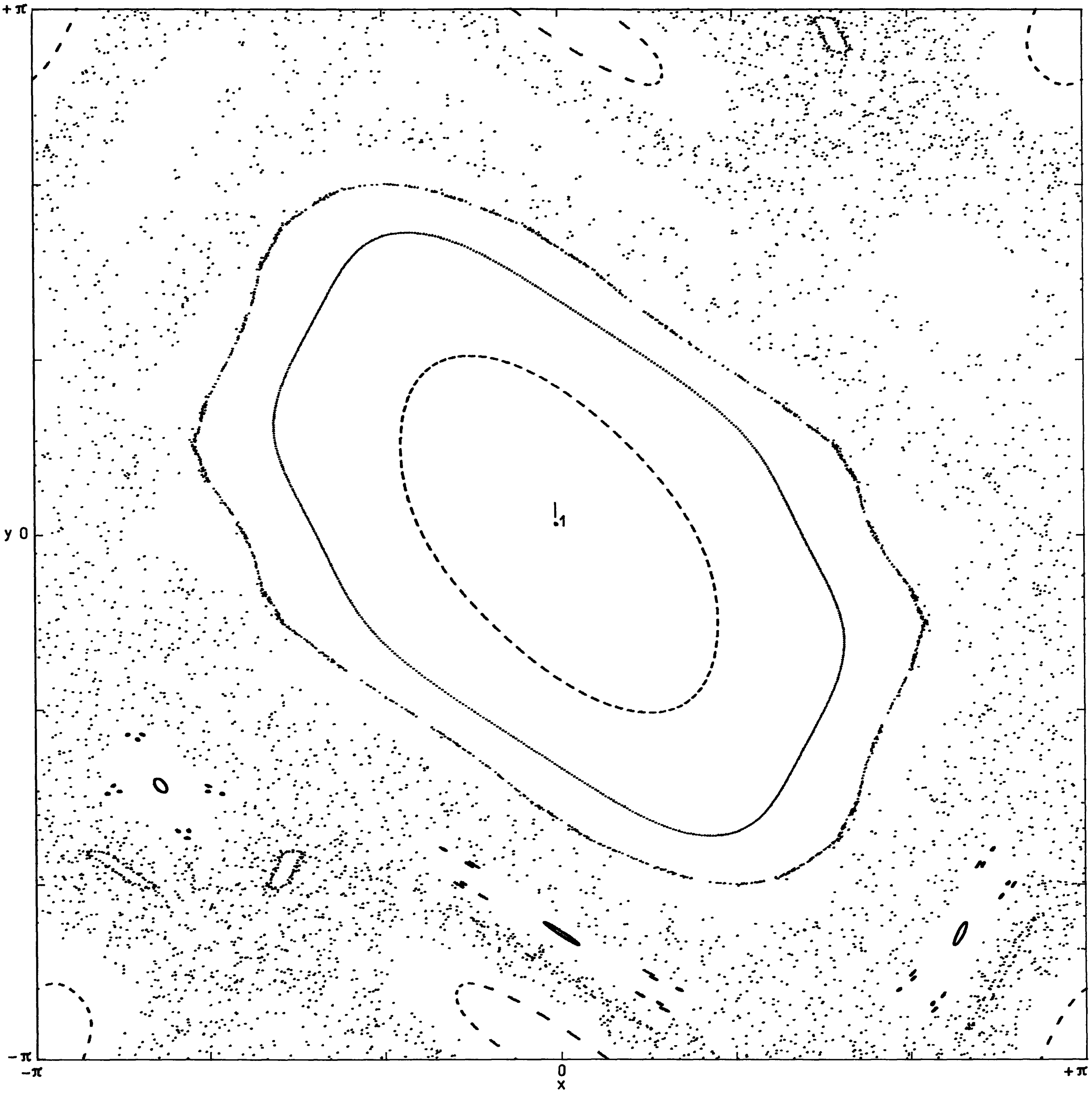

Fig. 8. Typical sets of points for nine initial conditions (x_0, y_0) and for $\lambda = 1.30$ obtained by repeated applications of the mapping T_0 [see Eq. (5)]. Each set has 1000 points

Table 4

m	Total number of cycles		Length	
	Random mapping	Numerical experiments	Random mapping average	Numerical experiments. Longest cycle.
300	11.98	592	45000	1394
400	12.56	768	80000	2058
500	13.01	986	125000	3932
600	13.37	1164	180000	2400
700	13.68	1364	245000	3218
800	13.95	1566	320000	3954

This table represents the results of T_0^* for $\lambda = 10.$ and different values of the discretisation m.

preserving mappings:

$$P_i \xrightarrow{R} Q_i \xrightarrow{S} P_{i+1}$$

$$(x_i, y_i) \quad (x_{i+1}, y_i) \quad (x_{i+1}, y_{i+1})$$

$$R\begin{cases} x' = x + y \\ y' = y \end{cases} \quad S\begin{cases} x' = x \\ y' = y - \lambda \sin x \,. \end{cases} \tag{7}$$

On Fig. 12, we see that to any set of points obtained by alternating applications of the mappings R and S, there corresponds another set of points that is symmetrical with respect to the x axis and described in the opposite direction.

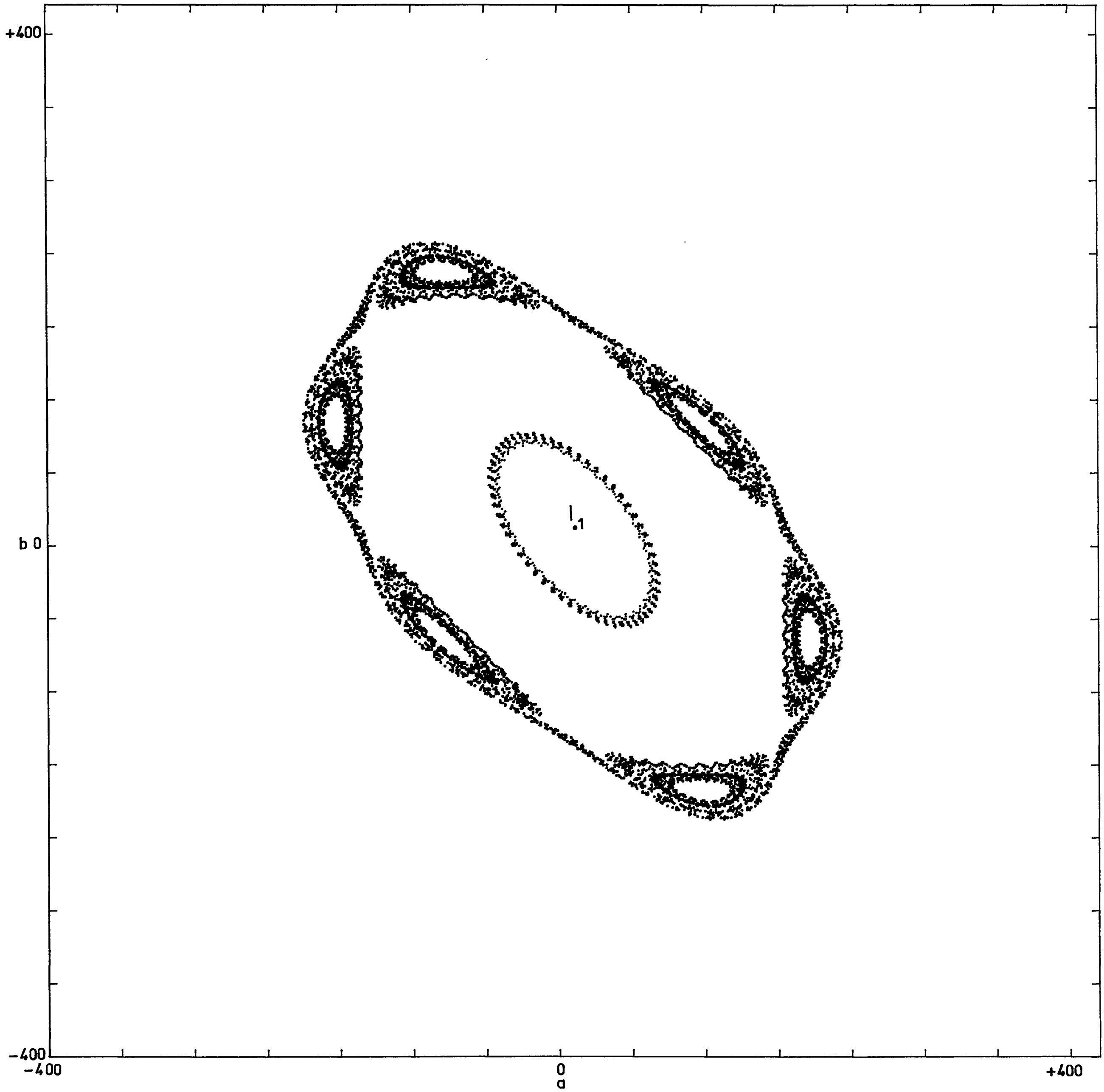

Fig. 9. Two sets of points obtained with the discrete mapping T_0^* [see Eq. (6)] for $\lambda = 1.30$ and $m = 800$

A similar symmetry exists with respect to the y axis. A complete analysis reveals the existence of seven cases of symmetry; they are schematically represented on Fig. 13.

In the case of the mapping T_0^* we write similarly:

$$R\begin{cases} a' = a + b \\ b' = b \end{cases} \quad S\begin{cases} a' = a \\ b' = b - \left[\dfrac{\lambda m}{2\pi} \sin \dfrac{2\pi}{m} a\right] \end{cases} \tag{8}$$

and the same symmetries exist. Because of these, the mapping T_0^* is degenerate and its properties are different from those of a random mapping. In particular the cycles tend to close back much earlier. A full analysis, taking the symmetries into account but assuming randomness otherwise, gives results in very good agreement with the observed properties (Rannou, 1972); for instance the predicted total number of cycles for $m = 700$ is 1330 (against 1364 observed) and the predicted length of the longest cycle is 3383 (against 3218 observed).

In order to get rid of the symmetries, we note that (7) can be generalized to

$$R\begin{cases} x' = x + f(y) \\ y' = y \end{cases} \quad S\begin{cases} x' = x \\ y' = y + g(x). \end{cases} \tag{9}$$

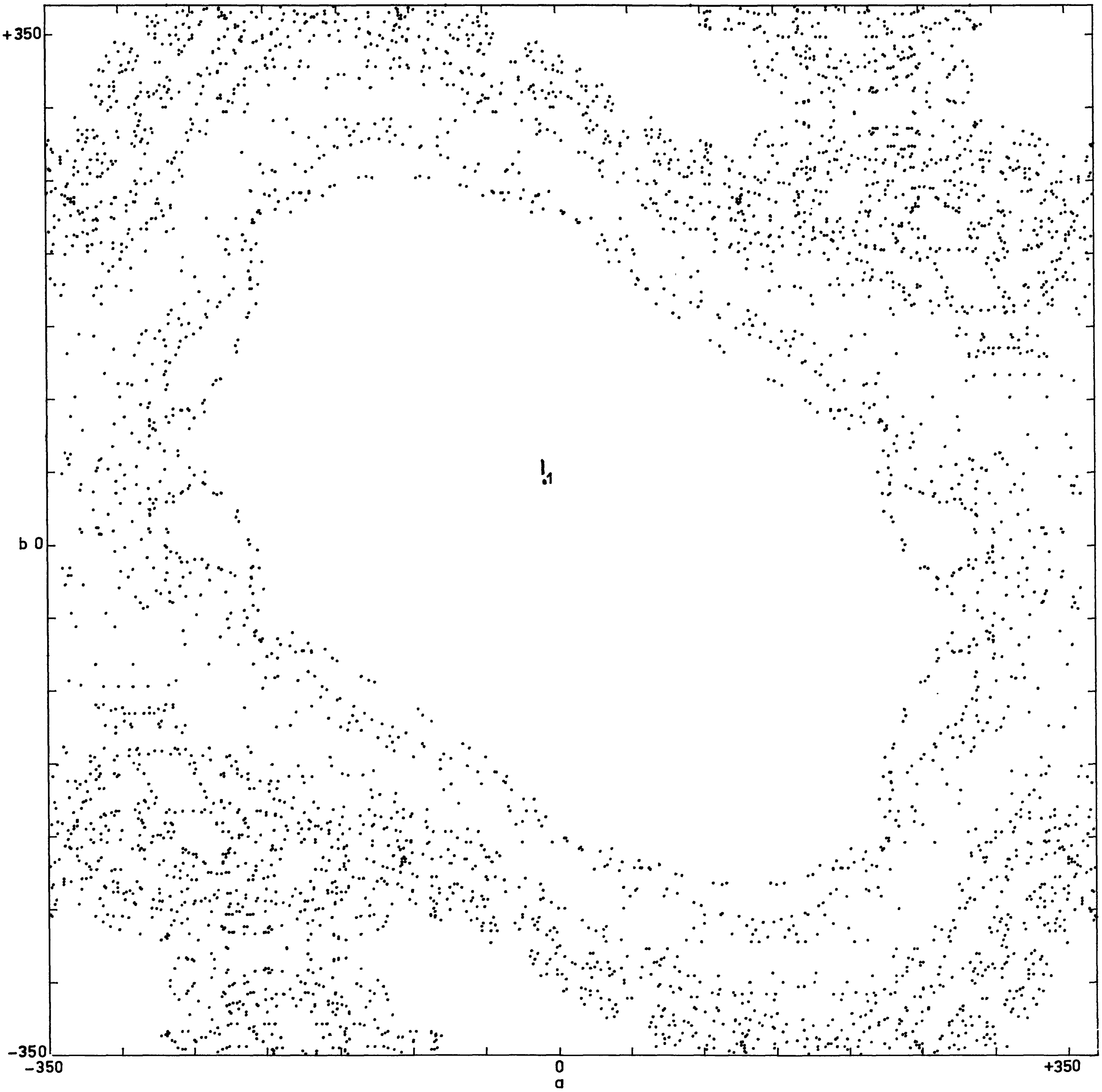

Fig. 10. A cycle obtained with T_0^* for $\lambda = 1.30$ and $m = 700$. The length of this cycle is 3878 points

The symmetries of T_0 result from the fact that f and g are added functions: $f(-y) = -f(y)$; $g(-x) = -g(x)$. We therefore modify the mapping T_0 by adding even terms that destroy the symmetries. These functions should be periodic with period 2π. Also it is convenient to leave the point $I(0, 0)$ invariant. These considerations led us to the mapping T given by Eq. (1).

VI. Conclusions

1. The figures obtained by repeated applications of a discrete mapping are generally similar to those obtained from a continuous mapping, or from the solution of the differential equations representing a mechanical system. See for instance Fig. 8 and 9. For Fig. 9 the discretisation is high ($m = 800$). But similar results are also found for a rough discretisation ($m = 200$). This encourages the view that conventional computer studies are not seriously affected by round-off errors, since m is then typically of the order of 10^8.

2. In the discrete case, the total number of accessible points is finite and equal to m^2. This has allowed us to define and study new quantities such as the total number of cycles or the length of a cycle.

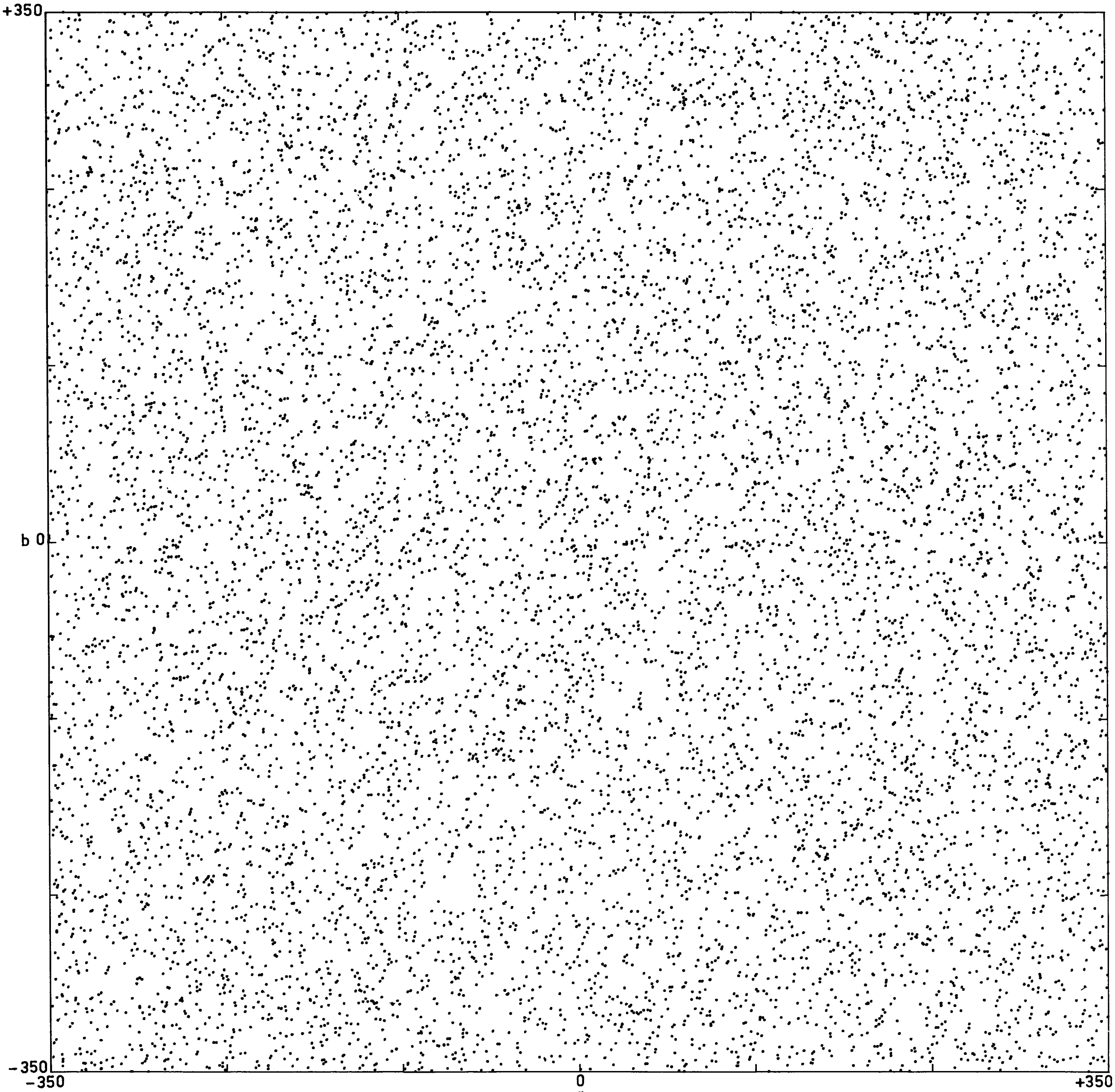

Fig. 11. A cycle of 3218 points obtained with T_0^* for $\lambda = 10$ and $m = 700$

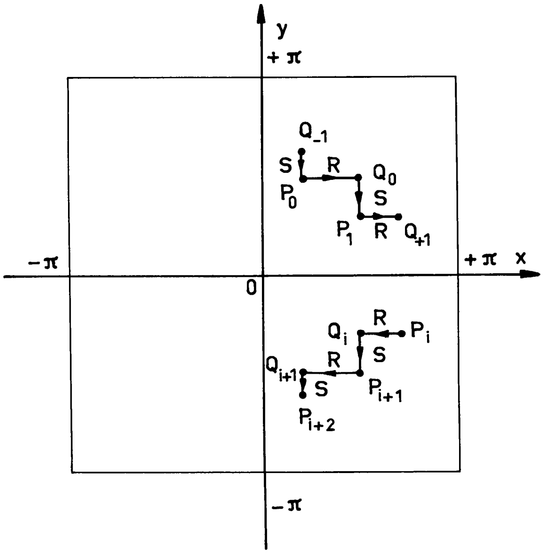

Fig. 12. Symmetry of the mapping T_0 with respect to the x axis

3. When $\lambda > 4$, i.e. when the two invariant points are unstable, the figures suggest a quasi-random behaviour. Accordingly, we have defined a "random mapping" for which some simple properties have been found. The observed properties agree remarkably well with the theoretical properties of the random mapping. Particularly the total number of cycles and the average length of a cycle are very similar to those of a random mapping (Section IV).

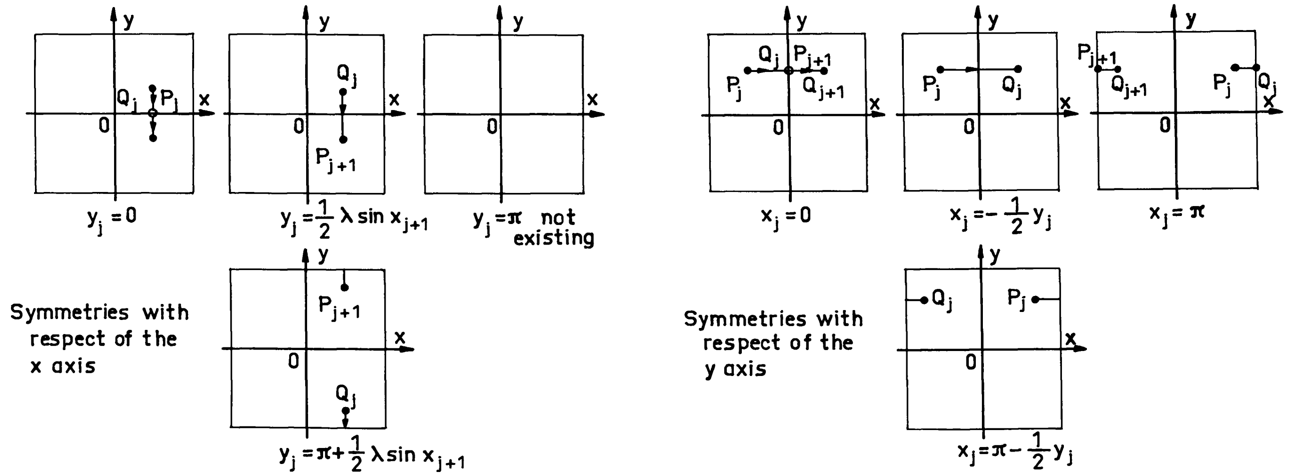

Fig. 13. The seven cases of symmetry of the mapping T_0

4. This study has shown, one the other hand, that one must be cautious in the choice of a representative mapping: certain symmetries can make the mapping degenerate (Section V).

5. In the discrete case, it is possible to study the "ergodic" part of the plane in a better way than with a continuous mapping. Figures 4 and 5 show that the points of one cycle in this region are distributed in a random way.

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