C. Phase and Energy (Group) Velocities The wave velocity v_r of the fields given by (4-18a) and (4-18b) in the direction $\boldsymbol{\beta}$ of travel is equal to the speed of light v. Since the wave is a plane wave to the direction $\boldsymbol{\beta}$ of travel, the planes over which the phase is constant (constant phase planes) are perpendicular to the direction $\boldsymbol{\beta}$ of wave travel. This is illustrated graphically in Figure 4-6. To maintain a constant phase (or to keep in step with a constant phase plane), a velocity equal to the speed of light must be maintained in the direction $\boldsymbol{\beta}$ of travel. This is referred to as the phase velocity v_{pr} along the direction $\boldsymbol{\beta}$ of travel. Since the energy also is being transported with the same speed, the energy velocity v_{er} in the direction $\boldsymbol{\beta}$ of travel is also equal to the speed of light. Thus

$$v_r = v_{pr} = v_{er} = v = \frac{1}{\sqrt{\mu\varepsilon}} \tag{4-22}$$

where

$v_r =$ wave velocity in the direction of wave travel
$v_{pr} =$ phase velocity in the direction of wave travel
$v_{er} =$ energy (group) velocity in the direction of wave travel
$v =$ speed of light

To keep in step with a constant phase plane of the wave of Figure 4-6, a velocity in the z direction equal to

$$v_{pz} = \frac{v}{\cos\theta_i} = \frac{1}{\sqrt{\mu\varepsilon}\cos\theta_i} \geq v \tag{4-23}$$

must be maintained. This is referred to as the phase velocity v_{pz} in the z direction, and it is greater than the speed of light. Since nothing travels with speeds greater than the speed of light, it must be remembered that this is a hypothetical velocity that must be maintained in order to

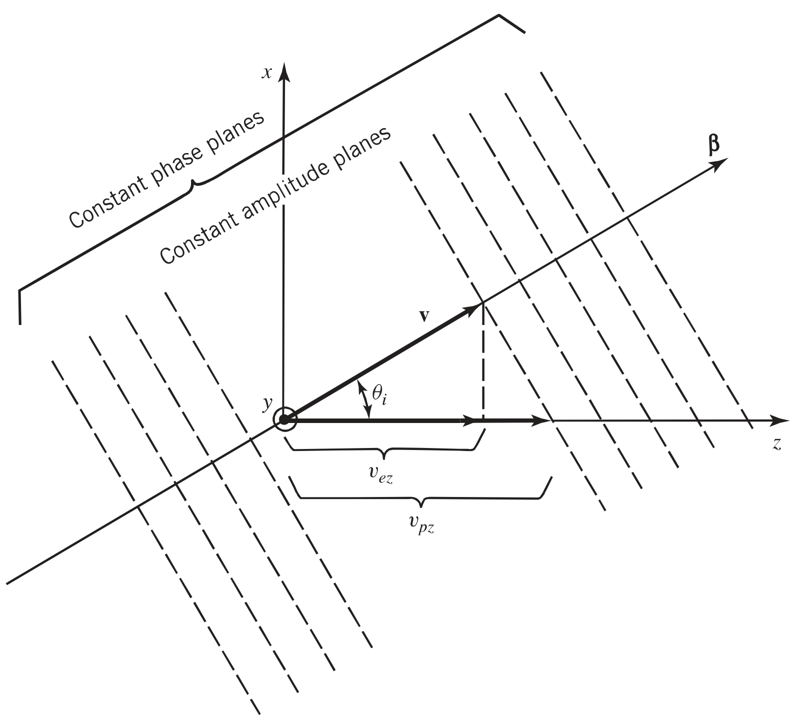

Figure 4-6 Phase and energy (group) velocities of a uniform plane wave.

keep in step with a constant phase plane of the wave that itself travels with the speed of light in the direction $\boldsymbol{\beta}$ of travel. The phase velocities of (4-22) and (4-23) can be obtained, respectively, by using (4-19c) and (4-19d). These are left as end-of-chapter exercises for the reader.

Whereas a velocity greater than the speed of light must be maintained in the z direction to keep in step with a constant phase plane of Figure 4-6, the energy is transported in the z direction with a velocity that is equal to

$$v_{ez} = v \cos\theta_i = \frac{\cos\theta_i}{\sqrt{\mu\varepsilon}} \leq v \tag{4-24}$$

This is referred to as the energy (group) velocity v_{ez} in the z direction, and it is equal to or smaller than the speed of light. Graphically this is illustrated in Figure 4-6.

For any wave, the product of the phase and energy velocities in any direction must be equal to the speed of light squared or

$$v_{pr} v_{er} = v_{pz} v_{ez} = v^2 = \frac{1}{\mu\varepsilon} \tag{4-25}$$

This obviously is satisfied by the previously derived results.

The energy velocity of (4-24) can be derived using (4-18a) and (4-18b) along with the definition (4-9). This is left for the reader as an end-of-chapter exercise.

Since the fields of (4-18a) and (4-18b) form a uniform plane wave, the planes over which the amplitude is maintained constant are also constant planes that are perpendicular to the direction $\boldsymbol{\beta}$ of travel. These are illustrated in Figure 4-6 and coincide with the constant phase planes. For other types of waves, the constant phase and amplitude planes do not in general coincide.

D. Power and Energy Densities The average power density associated with the fields of (4-18a) and (4-18b) that travel in the $\boldsymbol{\beta}^+$ direction is given by

$$\begin{aligned}
\left(\mathbf{S}_{\mathrm{av}}^{+}\right)_r &= \frac{1}{2}\mathrm{Re}\left[\left(\mathbf{E}^{+}\right) \times \left(\mathbf{H}^{+}\right)^{*}\right] \\
&= \frac{1}{2}\mathrm{Re}\left[E_0^{+}\left(\hat{\mathbf{a}}_x \cos\theta_i - \hat{\mathbf{a}}_z \sin\theta_i\right) e^{-j\beta(x\sin\theta_i + z\cos\theta_i)}\right. \\
&\qquad\qquad \left. \times \hat{\mathbf{a}}_y \frac{E_0^{+*}}{\eta} e^{+j\beta(x\sin\theta_i + z\cos\theta_i)}\right] \\
\left(\mathbf{S}_{\mathrm{av}}^{+}\right)_r &= \left(\hat{\mathbf{a}}_x \sin\theta_i + \hat{\mathbf{a}}_z \cos\theta_i\right)\frac{|E_0^{+}|^2}{2\eta} = \hat{\mathbf{a}}_r \frac{|E_0^{+}|^2}{2\eta} = \hat{\mathbf{a}}_x \left(S_{\mathrm{av}}^{+}\right)_x + \hat{\mathbf{a}}_z \left(S_{\mathrm{av}}^{+}\right)_z
\end{aligned} \tag{4-26}$$

where

$$\left(S_{\mathrm{av}}^{+}\right)_x = \sin\theta_i \frac{|E_0^{+}|^2}{2\eta} = \sin\theta_i \left(S_{\mathrm{av}}^{+}\right)_r \tag{4-26a}$$

$$\left(S_{\mathrm{av}}^{+}\right)_z = \cos\theta_i \frac{|E_0^{+}|^2}{2\eta} = \cos\theta_i \left(S_{\mathrm{av}}^{+}\right)_r \tag{4-26b}$$

$(S_{\mathrm{av}}^{+})_r$ represents the average power density along the principal $\boldsymbol{\beta}^+$ direction of travel and $(S_{\mathrm{av}}^{+})_x$ and $(S_{\mathrm{av}}^{+})_z$ represent the directional power densities of the wave in the $+x$ and $+z$ directions, respectively. Similar expressions can be derived for the wave that travels along the $\boldsymbol{\beta}^-$ direction.