Generalized Eigenfunctions

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1 Introduction

We will consider linear operators in complex Hilbert Spaces. To fix notation, generally we will denote any Hilbert Space by $\mathcal{H}$, possibly with a subindex. On $\mathcal{H}$, denote the inner product by $(,)$. For any operator T on $\mathcal{H}$, by $\mathcal{D}(T)$ we mean the domain of T and by T^* the adjoint of T. We will always deal with operators having dense domain, i.e. $\overline{\mathcal{D}(T)} = \mathcal{H}$. By $\sigma(T)$, we denote the spectrum of T. For a detailed introduction of the above definitions and notations, one can refer to The Schrödinger Equation by F.A. Berezin and M.A. Shubin. Now let us give a useful definition of an eigenvalue of a self adjoint operator.

Definition 1.1 *If T is a self-adjoint operator on $\mathcal{H}$ (i.e. $T = T^*$), then $f \in \mathcal{H}$ is an eigenvector of T with eigenvalue λ if and only if for any $g \in \mathcal{D}(T)$, we have $(f, Tg) = \lambda(f, g)$.*

Notice the advantage of this definition than the usual definition of eigenvalue. In the above definition, f does not have to be in the domain of T. Any eigenvalue of an operator T is always in the spectrum of T. Now let us show with an example that an operator T can have an infinite spectrum while the set of eigenvalues of T is empty.

Example 1.2 *Let $\mathcal{H} = L^2(\mathbb{R}, dx)$, and let T be the operator of multiplication by x in $\mathcal{H}$. Then any $\lambda \in \mathbb{R}$ is in the spectrum of T, because the operator of multiplication by $\frac{1}{x-\lambda}$ is an unbounded operator in $\mathcal{H}$. But such a λ can not be an eigenvalue, because if $Tf = \lambda f$ for some non-zero $f \in \mathcal{H}$, $f(x) \cdot (x - \lambda) \equiv 0$, which means that $f(x) = 0$ almost everywhere, contradiction. So T has no eigenvalue.*

One can notice that from the above argument such an f is in fact the $\delta-$function. But we know that $\delta-$function is not in $L^2(\mathbb{R}, dx)$.

We will show that for some special kind of operators, with some extension $\mathcal{H}_-$ of $\mathcal{H}$, we can find some elements in $\mathcal{H}_-$ acting like eigenvectors of an operator $A \in \mathcal{H}$, namely Generalized Eigenvectors. For this we need to define Hilbert-Schmidt operators and some properties of these operators in the next section.

2 Hilbert-Schmidt Operators

Definition 2.1 *A bounded linear operator $K : \mathcal{H}_1 \to \mathcal{H}_2$ is a Hilbert-Schmidt operator if for an orthonormal basis $\{e_\alpha\}$ of $\mathcal{H}_1$ the sum $\sum_\alpha ||Ke_\alpha||^2$ is finite. When this sum is finite, we define*

$$||K||_2^2 = \sum_\alpha ||Ke_\alpha||^2$$

as the Hilbert-Schmidt norm of K.

Remark 2.2

1. *If K is a Hilbert-Schmidt operator, $||K||_2$ is well defined.*
2. *If K is a Hilbert-Schmidt operator, K^* is also Hilbert-Schmidt and $||K^*||_2 = ||K||_2$.*
3. *For any Hilbert-Schmidt operator K, $||K|| \le ||K||_2$.*
4. *For any separable Hilbert Space, there exists a Hilbert-Schmidt operator such that $K = K^*$ and $\ker K = 0$.*

Proof 2.3

1. *Let $\{f_\alpha\}$ be an orthonormal basis in $\mathcal{H}_2$. Then since $||Ke_\alpha||^2 = \sum_\beta |(Ke_\alpha, f_\beta)|^2$, we have*

$$||K||_2^2 = \sum_\alpha ||Ke_\alpha||^2 = \sum_{\alpha,\beta} |(e_\alpha, K^* f_\beta)|^2 = \sum_\beta ||K^* f_\beta||^2 = ||K^*||_2^2$$

which proves that the HS-Norm is independent of basis in $\mathcal{H}_1$.

2. *Given in (1).*
3. *Let $a = \sum_\alpha a_\alpha e_\alpha \in \mathcal{H}_1$. Then*

$$||Ka||^2 = ||\sum_\alpha a_\alpha K e_\alpha||^2 \le \left(\sum_\alpha |a_\alpha| ||Ke_\alpha||\right)^2$$
$$\le \sum_\beta |a_\beta|^2 \cdot \sum_\alpha ||Ke_\alpha||^2 = ||K||_2^2 ||a||^2,$$

i.e. $||Ka|| \le ||K||_2 ||a||$ which proves (3).

4. *Let $\{e_i\}$ be an orthonormal basis of $\mathcal{H}$, $\{\lambda_i\}$ be an l_2 sequence with all $\lambda_i > 0$. Then for any $a = \sum_i a_i e_i \in \mathcal{H}$, define $Ka = \sum_i \lambda_i a_i e_i$. One can check that this operator K satisfies all the required properties.*

Next proposition describes the form of Hilbert-Schmidt operators between two L^2- spaces.

Proposition 2.4 *An operator $K : L^2(M_1, d\mu_1) \to L^2(M_2, d\mu_2)$ is a Hilbert-Schmidt operator if and only if there exists $\mathcal{K}(m_2, m_1) \in L^2(M_2 \times M_1, d\mu_2 \times d\mu_1)$ such that*

$$Ka(m_2) = \int_{M_1} \mathcal{K}(m_2, m_1) \cdot a(m_1) d\mu_1$$

where the function $\mathcal{K}(m_2, m_1)$ is uniquely defined except on a set of zero measure (in $d\mu_2 \times d\mu_1$) and

$$||K||_2^2 = \int \int |\mathcal{K}(m_2, m_1)|^2 d\mu_2 d\mu_1.$$

Proof 2.5 *Let $\mathcal{H}_1 = L^2(M_1, d\mu_1)$ and $\mathcal{H}_2 = L^2(M_2, d\mu_2)$ and let $\{e_\alpha(m_1)\}$ and $\{f_\beta(m_2)\}$ be basis of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively.*
Now first let us show that $\{\overline{e_\alpha(m_1)} f_\beta(m_2)\}$ constitute a complete orthonormal system in $L^2(M_2 \times M_1)$; for this take any $F \in L^2(M_2 \times M_1)$, if

$$\int \int F(m_2, m_1) \overline{e_\alpha(m_1)} f_\beta(m_2) d\mu_1 d\mu_2 = 0$$

then by Fubini's theorem for almost all m_1, $\int_{M_2} F(m_2, m_1) f_\beta(m_2) d\mu_2 = 0$ so for any m_1, for almost all m_2, $F(m_2, m_1) = 0$, i.e. F is almost everywhere 0, so $\{\overline{e_\alpha(m_1)} f_\beta(m_2)\}$ is a basis for $L^2(M_2 \times M_1)$.

Now assume that K is a Hilbert-Schmidt Operator,

$$Ka = K\left(\sum_\alpha (a, e_\alpha) e_\alpha\right) = \sum_{\alpha,\beta} (a, e_\alpha)(Ke_\alpha, f_\beta) f_\beta$$

$$\text{and } ||K||_2^2 = \sum_{\alpha,\beta} |(Ke_\alpha, f_\beta)|^2 < +\infty.$$

Then define $\mathcal{K}(m_2, m_1) = \sum_{\alpha,\beta} (Ke_\alpha, f_\beta) \overline{e_\alpha(m_1)} f_\beta(m_2)$. Then clearly $\mathcal{K}(m_2, m_1) \in L^2(M_2 x M_1)$, and $||K||_2^2 = ||\mathcal{K}(m_2, m_1)||_{L^2(M_2 x M_1)}$. Conversely if $Ka(m_2) = \int_{M_1} \mathcal{K}(m_2, m_1) a(m_1) d\mu_1$, then

$$\mathcal{K}(m_2, m_1) = \sum_{\alpha,\beta} c_{\alpha\beta} f_\beta(m_2) \overline{e_\alpha(m_1)},$$

then since $\sum_{\alpha,\beta} |c_{\alpha\beta}|^2 < +\infty$ and $Ke_\alpha = \sum_\beta c_{\alpha\beta} f_\beta$, we get $||K||_2^2 = \sum_{\alpha,\beta} |c_{\alpha\beta}|^2$ which proves that K is a Hilbert-Schmidt Operator.

3 Hilbert-Schmidt Riggings

Let $K : \mathcal{H} \to \mathcal{H}$ be a Hilbert-Schmidt Operator such that $\ker K = \ker K^* = 0$. Then define $\mathcal{H}_+ = K\mathcal{H}$, i.e. $\mathcal{H}_+$ is the image of the operator K. Then define

$||.||_+$ on $\mathcal{H}_+$. So if $Kf = u$, $||u||_+ = ||K^{-1}u|| = ||f||$. Then $\mathcal{H}_+$ is Banach Space in $||.||_+-$norm. Now define $\mathcal{H}_- := \{$Continuous, anti-linear functionals on$\mathcal{H}_+\}$. So if $\alpha \in \mathcal{H}_-$, then α satisfies the following:

- $|\alpha(u)| \leq C_\alpha ||u||_+, \ \forall u \in \mathcal{H}_+ (continuity)$.
- $\alpha(\lambda_1 u + \lambda_2 v) = \overline{\lambda_1}\alpha(u) + \overline{\lambda_2}\alpha(v)(anti-linearity.$

From now on for $\alpha \in \mathcal{H}_-$ and $u \in \mathcal{H}_+$ by (α, u) we will mean $\alpha(u)$. Note that for any $f \in \mathcal{H}$, we can define a continuous anti-linear functional corresponding to f as $l_f(u) = (f, u)$ which implies that $\mathcal{H} \subset \mathcal{H}_-$ and note that $|l_f(u)| \leq ||f|| \cdot ||u|| \leq C \cdot ||u||_+$. Then setting $(u, \alpha) := \overline{(\alpha, u)}$ gives an extension of the inner product on $\mathcal{H}$. So we get a triple of spaces

$$\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-,$$

which is called a Hilbert-Schmidt Rigging.

Proposition 3.1 *$\mathcal{H}_-$ is obtained from $\mathcal{H}$ by taking the completion with respect to norm defined by*

$$(f, g)_- = (K^* f, K^* g), \ f, g \in \mathcal{H}.$$

Proof 3.2 *We need to show that $||h||_- = ||K^* h|| \ \forall h \in \mathcal{H}$ and $\mathcal{H}$ is dense in $\mathcal{H}_-$ in the topology of $\mathcal{H}_-$. To show this; letting $Kf = h_+$,*

$$||h||_- = \sup_{||h_+||_+=1} |(h, h_+)| = \sup_{||f||=1} |(h, Kf)| = \sup_{||f||=1} |(K^* h, f)| = ||K^* h||.$$

To show denseness of $\mathcal{H}$ in $\mathcal{H}_-$, extend $K^* : \mathcal{H}_- \to \mathcal{H}$ as an isometry, i.e. $\forall h_1, h_2 \in \mathcal{H}$, $(K^* h_1, K^* h_2)_- = (h_1, h_2)$. If $\mathcal{H}$ is not dense in $\mathcal{H}_-$, there exists a non-zero $h \in \mathcal{H}_-$ such that $(h, f)_- = 0 \ \forall f \in \mathcal{H}$. Let $K^* h = h_1 \in \mathcal{H}$, then $(K^* h, K^* f) = (h, f)_- = 0$ so $(h_1, K^* f) = (K h_1, f) = 0 \ \forall f \in \mathcal{H}$ then $K h_1 = 0$, contradictng the fact that K is injective.

4 Generalized Eigenfunctions

Definition 4.1 *Let M be a measure space, A a self-adjoint operator and $\phi(m) : M \to \mathcal{H}_-$ a vector valued function taking values in $\mathcal{H}_-$, rigging of a Hilbert Space $\mathcal{H}$. Then $\phi(m)$ is called a complete system of generalized eigenvectors of A if;*

1. *$\forall h_+ \in \mathcal{H}_+$, the function $m \mapsto (h_+, \phi(m))$ is in $L^2(M, d\mu)$.*
2. *We can extend the map $h_+ \mapsto (h_+, \phi(.))$ to a unitary operator $U : \mathcal{H} \to L^2(M, d\mu)$.*
3. *There exists a real valued function $a = a(m)$ that is measurable and almost everywhere finite on M and is such that $A = U^{-1}\hat{a}U$ where $\hat{a}$ is the multiplication operator by he function $a = a(m)$ in $L^2(M, d\mu)$.*

Now we can state the main theorem;

Theorem 4.2 *let $\mathcal{H}$ be a Hilbert Space. Given a self-adjoint operator A in $\mathcal{H}$ and a $Hilbert - Schmidt$ rigging of $\mathcal{H}$, there exists a complete system of generalized eigenfunctions of A.*

Proof 4.3 *Using spectral theorem we can think of A as the multiplication operator by $a(m)$ in $L^2(M, d\mu)$. Again using the same theorem we can think of $\mathcal{H}$ as $L^2(M, d\mu)$. Let K be the Hilbert-Schmidt operator used in constructing the Hilbert-Schmidt rigging of $\mathcal{H}$ or equivalently of $L^2(M, d\mu)$. Now taking as $\mathcal{H} = L^2(M, d\mu)$ and $A = a(m)$, since K is a Hilbert-Schmidt operator on an L^2 space, by proposition 2.4 there exists $\mathcal{K}(m_1, m_2) \in L^2(M \times M, d\mu \times d\mu)$ such that for any $f \in L^2(M, d\mu)$, $Kf(m) = \int_M \mathcal{K}(m, m_1) f(m_1) d\mu(m_1)$.*
Now we will take the $\delta-$function at m as the functional $\phi(m)$ on $\mathcal{H}_+$.
Define $\phi(m) := \delta_\lambda(m) = \delta(m - \lambda)$, the delta function at m, so

$$(f, \phi(m)) = f(m), \quad f \in \mathcal{H}_+.$$

Then assuming $Kg = f$ for some $g \in \mathcal{H}$,

$$(f, \phi(m)) = (Kg, \phi(m)) = \int_M \mathcal{K}(m, m_1) g(m_1) d\mu(m_1).$$

Now $\phi(m)$ is defined for almost all m, since;

$$||\phi(m)||_-^2 = \sup_{||g||=1, g \in \mathcal{H}} |(Kg, \phi(m)|^2 = \sup_{||g||=1} \left(\int_M \mathcal{K}(m, m_1) g(m_1) d\mu(m_1) \right)^2$$

$$\leq \int_M |\mathcal{K}(m, m_1)|^2 \, d\mu(m_1) < +\infty$$

for almost all m, where the last inequality follows from Fubini's theorem. Now taking $g = \frac{\mathcal{K}(m, m_1)}{||\mathcal{K}||}$, we have $\phi(m) \geq \int_M |\mathcal{K}(m, m_1)|^2 \, d\mu(m_1)$ which implies that

$$||\phi(m)||_-^2 = \int_M (\mathcal{K}(m, m_1))^2 \, d\mu(m_1).$$

so $\phi(m)$ is defined for almost all m.

Example 4.4 *Returning back to example 1.2, for the multiplication operator by x on $L^2(\mathbb{R}.dx)$, applying the construction given in the above proof one can see that $\phi(m) = \delta(m)$ and what is more we have the following equality:*

$$(xf(x), \phi(\lambda)) = \lambda \, (f(x), \phi(\lambda))$$

which means that $\delta(\lambda)$ is a (generalized) eigenfuntion of the multiplication operator corresponding to eigenvalue λ for any $\lambda \in \mathbb{R}$.