The development of the method of exhaustion beyond the point to which Archimedes carried it had to wait nearly eighteen centuries until the use of algebraic symbols and techniques became a standard part of mathematics. The elementary algebra that is familiar to most high-school students today was completely unknown in Archimedes' time, and it would have been next to impossible to extend his method to any general class of regions without some convenient way of expressing rather lengthy calculations in a compact and simplified form.

A slow but revolutionary change in the development of mathematical notations began in the 16th Century A.D. The cumbersome system of Roman numerals was gradually displaced by the Hindu-Arabie characters used today, the symbols + and – were introduced for the first time, and the advantages of the decimal notation began to be recognized. During this same period, the brilliant successes of the Italian mathematicians Tartaglia,

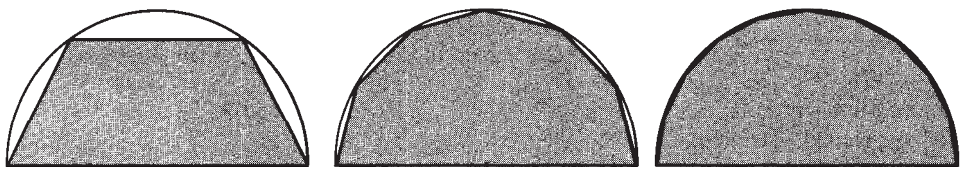

FIGURE 1.2 The method of exhaustion applied to a semicircular region.

Cardano, and Ferrari in finding algebraic solutions of cubic and quartic equations stimulated a great deal of activity in mathematics and encouraged the growth and acceptance of a new and superior algebraic language. With the widespread introduction of well-chosen algebraic symbols, interest was revived in the ancient method of exhaustion and a large number of fragmentary results were discovered in the 16th Century by such pioneers as Cavalieri, Toricelli, Roberval, Fermat, Pascal, and Wallis.

Gradually the method of exhaustion was transformed into the subject now called integral calculus, a new and powerful discipline with a large variety of applications, not only to geometrical problems concerned with areas and volumes but also to problems in other sciences. This branch of mathematics, which retained some of the original features of the method of exhaustion, received its biggest impetus in the 17th Century, largely due to the efforts of Isaac Newton (1642-1727) and Gottfried Leibniz (1646–1716), and its development continued well into the 19th Century before the subject was put on a firm mathematical basis by such men as Augustin-Louis Cauchy (1789-1857) and Bernhard Riemann (1826-1866). Further refinements and extensions of the theory are still being carried out in contemporary mathematics.

I1.3 The method of exhaustion for the area of a parabolic segment

Before we proceed to a systematic treatment of integral calculus, it will be instructive to apply the method of exhaustion directly to one of the special figures treated by Archimedes himself. The region in question is shown in Figure 1.3 and can be described as follows: If we choose an arbitrary point on the base of this figure and denote its distance from 0 by x, then the vertical distance from this point to the curve is x^2. In particular, if the length of the base itself is b, the altitude of the figure is b^2. The vertical distance from x to the curve is called the "ordinate" at x. The curve itself is an example of what is known

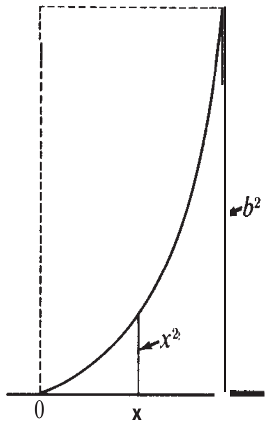

Figure 1.3 A parabolic segment.

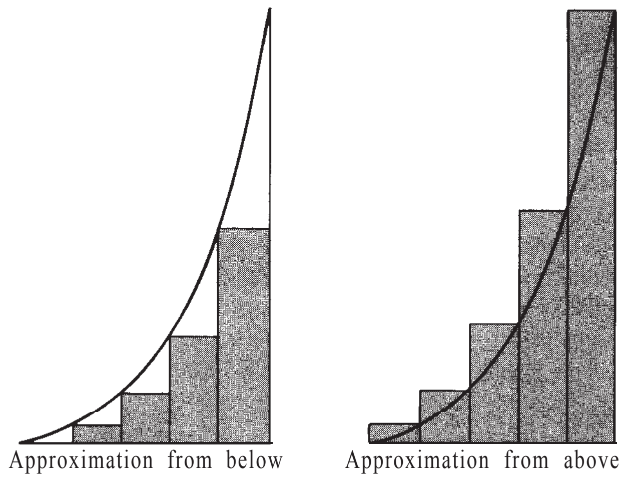

Figure 1.4

as a ***parabola.*** The region bounded by it and the two line segments is called a ***parabolic segment.***

This figure may be enclosed in a rectangle of base b and altitude b^2, as shown in Figure 1.3. Examination of the figure suggests that the area of the parabolic segment is less than half the area of the rectangle. Archimedes made the surprising discovery that the area of the parabolic segment is exactly *one-third* that of the rectangle; that is to say, $A = b^3/3$, where A denotes the area of the parabolic segment. We shall show presently how to arrive at this result.

It should be pointed out that the parabolic segment in Figure 1.3 is not shown exactly as Archimedes drew it and the details that follow are not exactly the same as those used by him.

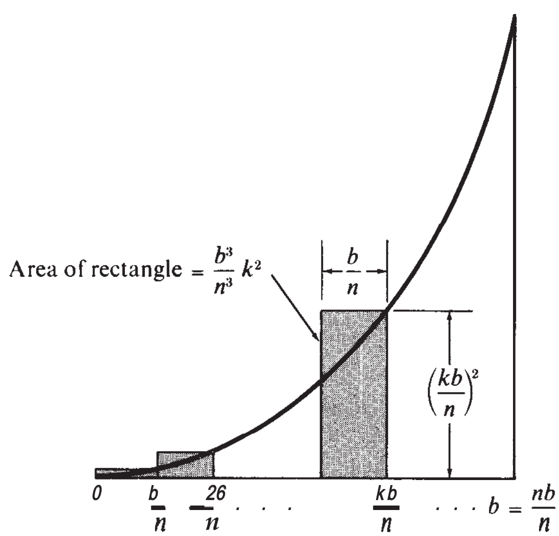

Figure 1.5 Calculation of the area of a parabolic segment.

Nevertheless, the essential *ideas* are those of Archimedes; what is presented here is the method of exhaustion in modern notation.

The method is simply this: We slice the figure into a number of strips and obtain two approximations to the region, one from below and one from above, by using two sets of rectangles as illustrated in Figure 1.4. (We use rectangles rather than arbitrary polygons to simplify the computations.) The area of the parabolic segment is larger than the total area of the inner rectangles but smaller than that of the outer rectangles.

If each strip is further subdivided to obtain a new approximation with a larger number of strips, the total area of the inner rectangles *increases,* whereas the total area of the outer rectangles *decreases.* Archimedes realized that an approximation to the area within any desired degree of accuracy could be obtained by simply taking enough strips.

Let us carry out the actual computations that are required in this case. For the sake of simplicity, we subdivide the base into n *equal* parts, each of length b/n (see Figure 1.5). The points of subdivision correspond to the following values of x:

$$0, \frac{b}{n}, \frac{2b}{n}, \frac{3b}{n}, \ldots, \frac{(n-1)b}{n}, \frac{nb}{n} = b$$

A typical point of subdivision corresponds to $x = kb/n$, where k takes the successive values $k = 0, 1, 2, 3, \ldots, n$. At each point kb/n we construct the outer rectangle of altitude $(kb/n)^2$ as illustrated in Figure 1.5. The area of this rectangle is the product of its base and altitude and is equal to

$$\left(\frac{b}{n}\right)\left(\frac{kb}{n}\right)^2 = \frac{b^3}{n^3}k^2.$$

Let us denote by S_n the sum of the areas of all the outer rectangles. Then since the kth rectangle has area $(b^3/n^3)k^2$, we obtain the formula

$$S_n = \frac{b^3}{n^3}(1^2 + 2^2 + 3^2 + \cdots + n^2). \tag{I.1}$$

In the same way we obtain a formula for the sum s_n of all the inner rectangles:

$$s_n = \frac{b^3}{n^3}[1^2 + 2^2 + 3^2 + \cdots + (n-1)^2]. \tag{I.2}$$

This brings us to a very important stage in the calculation. Notice that the factor multiplying b^3/n^3 in Equation (1.1) is the sum of the squares of the first n integers:

$$1^2 + 2^2 + \cdots + n^2.$$

[The corresponding factor in Equation (1.2) is similar except that the sum has only $n - 1$ terms.] For a large value of n, the computation of this sum by direct addition of its terms is tedious and inconvenient. Fortunately there is an interesting identity which makes it possible to evaluate this sum in a simpler way, namely,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}. \tag{I.3}$$

This identity is valid for every integer $n \geq 1$ and can be proved as follows: Start with the formula $(k + 1)^3 = k^3 + 3k^2 + 3k + 1$ and rewrite it in the form

$$3k^2 + 3k + 1 = (k+1)^3 - k^3.$$

Taking $k = 1, 2, \ldots, n - 1$, we get $n - 1$ formulas

$$3 \cdot 1^2 + 3 \cdot 1 + 1 = 2^3 - 1^3$$
$$3 \cdot 2^2 + 3 \cdot 2 + 1 = 3^3 - 2^3$$

$$3(n-1)^2 + 3(n-1) + 1 = n^3 - (n-1)^3.$$

When we add these formulas, all the terms on the right cancel except two and we obtain

$$3[1^2 + 2^2 + \cdots + (n-1)^2] + 3[1 + 2 + \cdots + (n-1)] + (n-1) = n^3 - 1^3.$$

The second sum on the left is the sum of terms in an arithmetic progression and it simplifies to $\frac{1}{2}n(n - 1)$. Therefore this last equation gives us

(I.4) $$1^2 + 2^2 + \cdots + (n-1)^2 = \frac{n^{-}}{-} - \frac{n^{-}}{-} + \frac{n}{-}.$$

Adding n^2 to both members, we obtain (1.3).

For our purposes, we do not need the exact expressions given in the right-hand members of (1.3) and (1.4). All we need are the two *inequalities*

(I.5) $$1^2 + 2^2 + \cdots + (n-1)" < \frac{n^3}{3} < 1^2 + 2^2 + \cdots + n^2$$

which are valid for every integer $n \geq 1$. These inequalities can de deduced easily as consequences of (1.3) and (I.4), or they can be proved directly by induction. (A proof by induction is given in Section 14.1.)

If we multiply both inequalities in (1.5) by b^3/n^3 and make use of (1.1) and (I.2), we obtain

(I.6) $$s_n < \frac{b^3}{3} < S_n$$

for every n. The inequalities in (1.6) tell us that $b^3/3$ is a number which lies between s_n and S_n for every n. We will now prove that $b^3/3$ is the *only* number which has this property. In other words, we assert that if A is any number which satisfies the inequalities

(1.7) $$s_n < A < S_n$$

for every positive integer n, then $A = b^3/3$. It is because of this fact that Archimedes concluded that the area of the parabolic segment is $b^3/3$.