A Disproval of Polignac's and Twin Primes Conjecture

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1 Introduction

The Polignac's conjecture states that there are infinite consecutive prime numbers p and p', such that $p - p' = 2k$ for $k = 1, 2, 3...$ where $k = 1$ is represents the special case of the twin primes conjecture [1].

The result presented here shows, by contradiction, that every consecutive prime gap of size $2k$ must be finite.

2 Proof

It is well known that there are infinitely many prime numbers [2], and that the gaps between prime numbers can be shown to be arbitrarily large[3]. Using these two facts, construct the following unbounded matrix A:

$$\begin{array}{c c}
 & \begin{array}{ccccccc} 2 & 4 & 6 & 8 & \dots & 2k & \dots \end{array} \\
\begin{array}{c} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_n \\ \vdots \end{array} &
\begin{pmatrix}
1 & 0 & 0 & 0 & \dots & 0 & \dots \\
1 & 0 & 0 & 0 & \dots & 0 & \dots \\
0 & 1 & 0 & 0 & \dots & 0 & \dots \\
\vdots & \vdots & \vdots & \ddots & \dots & \dots & \dots \\
0 & 0 & 0 & 0 & \dots & 1 & \dots \\
\vdots & \vdots & \vdots & \ddots & \dots & \dots & \dots
\end{pmatrix}
\end{array}$$

The row headers $p_1, p_2, p_3, ...$ represent consecutive prime numbers starting from $p_1 = 3$. Since there are infinitely many prime numbers, on any row n, corresponding to row header p_n, there exists a larger prime p_{n+1} such that $p_{n+1} - p_n = 2k$ for some k. On that row, a 1 is placed in the column with header $2k$ and a 0 everywhere else. So for example, the next prime number that occurs after $p_1 = 3$ is $p_2 = 5$; there is a gap of size 2 between these numbers and so a 1 is placed in column header 2 of row p_1 and 0 everywhere else.

Now, construct another unbounded matrix B:

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \cdots & \frac{1}{2^k} & \cdots \\ \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots & \frac{1}{2^{k-1}} & \cdots \\ \frac{1}{4} & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{4} & \cdots & \frac{1}{2^{k-2}} & \cdots \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 1 & \frac{1}{2} & \cdots & \frac{1}{2^{k-3}} & \cdots \\ \frac{1}{16} & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 1 & \cdots & \frac{1}{2^{k-4}} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2^k} & \frac{1}{2^{k-1}} & \frac{1}{2^{k-2}} & \frac{1}{2^{k-3}} & \frac{1}{2^{k-4}} & \cdots & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \end{pmatrix}$$

Here, $B_{i,i} = 1$ for all $i \geq 1$, $B_{i+j,i} = B_{i,i+j} = \frac{B_{i,i}}{2^j}$ for all $j \geq 1$.

Finally, construct the unbounded matrix $C = A - B$, such that $c_{i,j} = a_{i,j} - b_{i,j}$ for all $i, j \geq 1$.

2.1 The sum of matrix C by Rows

Consider the summation of the j^{th} row of matrix C

$$C_j = \sum C(j,:) = \sum A(j,:) - \sum B(j,:) \tag{1}$$

By construction, we know there exists only 1 non zero value equal to 1 in the j^{th} row of matrix A, therefore:

$$\sum A(j,:) = 1 \tag{2}$$

Due to the geometric series construction of matrix B, the summation of the j^{th} row can trivially be shown to converge to the following constant:

$$\sum B(j,:) = 2 + \sum_{k=1}^{j-1} \frac{1}{2^k} \tag{3}$$

Subtracing the two j^{th} rows yields the following:

$$C_j = \sum C(j,:) = -1 - \sum_{k=1}^{j-1} \frac{1}{2^j} \leq -1 \tag{4}$$

Since every row converges to a constant less than or equal to -1, the total summation of the matrix C diverges to:

$$C = \sum_{j=1}^{\infty} C_j = -\infty \tag{5}$$

2.2 The sum of matrix C by Columns

Now consider the summation of the k^{th} row of matrix C.

$$C_k = \sum C(:,k) = \sum A(:,k) - \sum B(:,k) \tag{6}$$

By construction, the number of occurences of $1's$ in column header $2k$ will exactly equal to the number of consecutive primes gaps of sizes $2k$.

If we assume that the Polignac's Conjecture is true, then the occurence of $1's$ that appear in any column header $2k$ should be infinite.

$$\sum A(:,k) = +\infty \tag{7}$$

Due to the geometric series construction of matrix B, the summation of the k^{th} row can trivially be shown to converge to the following constant:

$$\sum B(:,k) = 2 + \sum_{j=1}^{k-1} \frac{1}{2^j} \tag{8}$$

Subtraction of the two columns yields the following

$$C_k = \sum C(:,k) = +\infty - (2 + \sum_{j=1}^{k-1} \frac{1}{2^j}) = +\infty \tag{9}$$

Since every column diverges to positive infinity, the total summation of the matrix C diverges to:

$$C = \sum_{k=1}^{\infty} C_k = +\infty \tag{10}$$

This leads to a contradiction with the earlier result in Equation (5). Therefore the occurence of $1's$ cannot be infinite in all column headers $2k$, disproving Polignac's Conjecture.

Furthermore, even if only one column header had an infinite occurence of $1's$, the summation of matrix C columnwise will not diverge to negative infinity, therefore all column headers must be finite, which includes the case of $k = 1$, disproving the twin primes conjecture.

In order for the columnwise summations of C to diverge to negative infinity, there must exist an infinite number of column summations $C(:, k)$ that are negative. By construction, the summation of any column $B(:, k)$ is bounded on the interval (2,3). Therefore there must be an infinite number of column summations $A(:, k)$ less than 2. In conclusion, there must exist an infinite number of consecutive prime gaps of size $2k$, that occur at most twice.

References

[1] http://en.wikipedia.org/wiki/Polignac's_conjecture.

[2] A simple proof of infinite many primes can be found here http://primes.utm.edu/notes/proofs/infinite/euclids.html.

[3] A simple proof of arbitrary large gaps can be found here http://primes.utm.edu/notes/gaps.html.