AN APPROACH TO THE ù(x) FUNCTION IN NUMBER THEORY

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1.-Abstract

the ù(x) function in number theory gives the number of primes in an interval from 2 to x where x is a Real Number, exact formulae of this function are known but unfortunately they have several problems, or their computation is very tricky or you must know all the primes to compute it,

In this document we pretend to give a new approach to the function ù(x) in number theory by using the mathematical methods of Euler Transform of an analytic series and Laplace direct and inverse transform, i think this method can simplify the computation of the function ù(x) for number theory, making it be a formula valid for all x using simple mathematical expresions through an integral, also we will provide a transformation to calculate the sum over all primes of any given function f.also we will provide a method to calculate the caracteristic function of primes (we call it B(n)) that can indicate if a number is or not a prime

2.-Calculation of ù(x)

Towards the document we will make use of alternating series , we will use the Euler,s Transfom of alternating series to improve their convergence (we make the supposition this is valid) of the series , the series will have with coefficients $a_k>0$ for all n.

Given the identity:

$$\sum_p f(p) = -\int_0^{\infty} dx\, ù(x) \frac{df}{dx} \qquad ù(x) = 0 \ \text{iif} \ x < 2$$

Where the sum is taken to be over all primes and f is differentiable over all R substitutin f(x) =exp(-sx) with Re(s)>0 then we would have:

(2) $$\sum_p \exp(-sp) = -s\int_0^{\infty} dx \exp(-sx) ù(x)$$

Or remembering the de definition of a Laplace transform:

$$\underset{p}{\text{-}} \exp(\text{à}\ sp) = \text{à}\ sL(\text{ù}(x))$$

Now if we could calculate the sum on the left we could apply the Laplace inverse transform to get $\text{ù}(x)$ unfortunately the sum is hard to calculate so we will seetle for an aproximation of it by using the Euler´s transform of an analytical series:

$$\underset{n=0}{\mathbf{P}} (\text{à}\ 1)^n(\text{ù}(n)\ \text{à}\ \text{ù}(n\ \text{à}\ 1) + 1)Z^n$$

$$= \underset{p}{\text{-}} Z^p + 2Z^2 + \frac{1}{1+Z} \qquad ?Z? < 1$$

The function $\text{ù}(n)\ \text{à}\ \text{ù}(n\ \text{à}\ 1)$ takes the value 0 or 1 depending on if n is prime or composite, and $(-1)^p$ is –1 for all primes but 2 and we have that the terms $a_n>0$ for all n so we can apply an Euler´s transform to the series so we get:

$$\underset{n=0}{\mathbf{P}} (\text{à}\ 1)^n(\text{ù}(n)\ \text{à}\ \text{ù}(n\ \text{à}\ 1) + 1)Z^n$$

$$= \left(\frac{1}{1+z}\right)[a_0\ \text{à}\ \mathbf{4}a_0\,!\,(z) + \mathbf{4}^2a_0\,!\,(z)^2\ \text{à}\ \ldots +]$$

In this cae $a_0=1$

Where Ä is the finite-difference operator acting like that

$$\mathbf{4}a_k = a_{k+1}\ \text{à}\ a_k \qquad , \qquad !\,(z) = \frac{Z}{1+Z}$$

Setting Z=exp(-s) in our Euler´s transform we coulld obtain an approach to

$$\underset{p}{\text{-}} \exp(\text{à}\ sp) = \qquad 2\exp(\text{à}\ 2s)\ \text{à}\ \frac{1}{1+\exp(\text{à}\,s)}\ \text{à}$$

$$\underset{n=0}{\mathbf{P}} (\text{à}\ 1)^n(\text{ù}(n)\ \text{à}\ \text{ù}(n\ \text{à}\ 1) + 1)e^{\text{à}sn}$$

Where due to Euler,s transform we have equated the equality:

$$\sum_{n=0} (\text{à } 1)^n(\text{ù}(n) \text{ à ù}(n \text{ à } 1) + 1)e^{\text{à}sn}$$

$$= \frac{1}{1+\exp(\text{à}s)}[a_0 \text{ à } \mathbf{4}a_0E(s) + \mathbf{4}^2a_0E^2(s) \text{ à } \ldots +]$$

Where the function E(s) is defined like this:

$$E(s) = \frac{\exp(\text{à}s)}{1+\exp(\text{à}s)}$$

Basically the R(s) function is an approach to the sum over all primes of the exponential exp(-sp) and with (4) and (1) we could obtain the results:

$$\frac{R(s)}{s} = L(\text{ù}(x)) \qquad \text{ù}(x) = L^{\text{à}1}\left(\frac{R(s)}{s}\right)$$

Where L^{-1} is the inverse Laplace trasnform so we get:

$$\text{ù}(x) = \frac{1}{2\text{ù}i}\int_{c\text{à}i\mathbf{1}}^{c+i\mathbf{1}} dsR(s)\frac{Exp(sx)}{s}$$

And finally we have $\text{ù}(x)$

Where we have called R(s) to the function

$$R(s) = 2\exp(\text{à } 2s) \text{ à } \frac{1}{1+\exp(\text{à}s)} \text{ à } \frac{1}{1+\exp(\text{à}s)}[a_0 \text{ à } \mathbf{4}a_0E(s) + \mathbf{4}^2a_0E^2(s) \text{ à } \ldots +]$$

Another variant of the methods can be applied to calculate the sum for all primes of any series providing this series converges for example if we want to calculate the sum of all over primes for a series f(x) we take the series:

$$\sum_{n=0} (\text{à } 1)^n(\text{ù}(n) \text{ à ù}(n \text{ à } 1) + 1)f(n)Z^n$$

$$?Z? < A \quad A > 1$$

Now if we set Z=1 providing the series converges we would have the identities:

$$-\sum_{n=0}(-1)^n(\pi(n)-\pi(n-1)+1)f(n)$$

$$+\,2f(2)+\sum_{n=0}(-1)^n f(n) = \sum_p f(p)$$

For our purpose we will apply the Euler,s Transform to the alternating series, in this case we would have then:

$$\sum_{n=0}(-1)^n(\pi(n)-\pi(n-1)+1)f(n)$$

$$=(\tfrac{1}{2})[a_0-\Delta a_0(\tfrac{1}{2})+\Delta^2 a_0(\tfrac{1}{4})-\ldots]$$

Where the a_0 term refers to the first term in the series in this case $a_0=f(0)$

So we managed to calculate a (approximate) sum of the expresion $\sum_p f(p)$

The election of the Euler,s Transform is not arbitrary since it is used to improve the convergence in alternating series calculating only a few terms, we have only chosen an alternating series knowing that for any prime but p=2 $(-1)^p$ is always –1.
The main difficult in this method is that the coefficients will depend on the values of the function $\pi(n)$ wich is the function we want to calculate however for lower values of n they are easy to calculate (by simple inspection), so using Euler,s transform we improve the convergence and have a good approach using only a few coefficients of $\pi(n)$ with n<10 will be enough for many purposes.

Another use of the theory would be to calculate the b(n) function defined like this:

b(n)=0 iif n is composite b(n)=1 iif n is prime

Knowing that $B(n)=\pi(n)-\pi(n-1)$ and taking into account the previous results then we have the formula:

$$B(n) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds R(s)\frac{e^{sn}}{s}(1-e^{-s})$$

So we can also provide a good method to obtain all the primes simply by representing the function B(n) over n and watching when B(n)=1 expresed as the integral above.