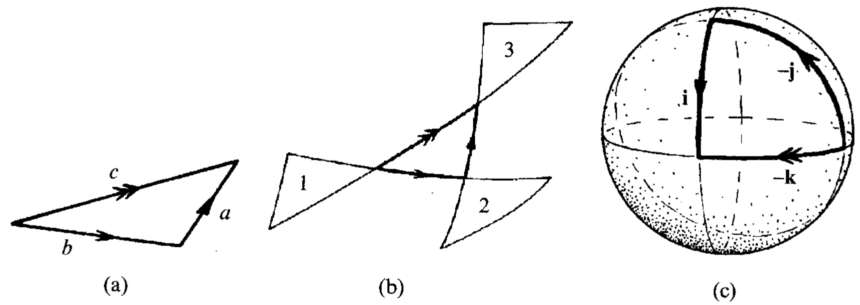

Fig. 11.4 (a) Translations in the Euclidean plane represented by oriented line segments. The double-arrowed segment represents the composition of the other two, by the triangle law. (b) For rotations in Euclidean 3-space, the segments are now great-circle arcs drawn on the unit sphere, each representing a rotation through *twice* the angle measured by the arc (about an axis perpendicular to its plane). To see why this works, reflect the triangle made by the arcs, in each vertex in turn. The first rotation takes triangle 1 into triangle 2, the second takes triangle 2 into triangle 3, and the composition takes triangle 1 into triangle 3. (c) The quaternionic relation $\mathbf{ij} = \mathbf{k}$ (in the form $\mathbf{i}(-\mathbf{j}) = -\mathbf{k}$), as a special case. The rotations are each through π, but represented by the half-angle $\frac{\pi}{2}$.

We can examine this in the particular situation that we considered above, and try to illustrate the quaternionic relation $\mathbf{ij} = \mathbf{k}$. The rotations described by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are each through an angle π. Thus, we use arc-lengths that are just half this angle, namely $\frac{1}{2}\pi$, in order to depict the 'triangle law'. This is fully illustrated in Fig. 11.4c (in the form $\mathbf{i}(-\mathbf{j}) = -\mathbf{k}$, for clarity). We can also see the relation $\mathbf{i}^2 = -1$ as illustrated by the fact that a great circle arc, of length π, stretching from a point on the sphere to its *antipodal* point (depicting '-1') is essentially different from an arc of zero length or of length 2π, despite the fact that each represents a rotation of the sphere that restores it to its original position. The 'vector arc' description correctly represents the rotations of a 'spinorial object'.

11.5 Clifford algebras

To proceed to higher dimensions and to the idea of a Clifford algebra, we must consider what the analogue of a 'rotation about an axis' must be. In n dimensions, the basic such rotation has an 'axis' which is an $(n-2)$-dimensional space, rather than just the 1-dimensional line-axis that we get for ordinary 3-dimensional rotations. But apart from this, a rotation about an $(n-2)$-dimensional axis is similar to the familiar case of an

ordinary 3-dimensional rotation about a 1-dimensional axis in that the rotation is completely determined by the direction of this axis and by the amount of the angle of the rotation. Again we have spinorial objects with the property that, if such an object is continuously rotated through the angle 2π, then it is not restored to its original state but to what we consider to be the 'negative' of that state. A rotation through 4π always does restore such an object to its original state.

There is, however, a 'new ingredient', alluded to above: that in dimension higher than 3, it is not true that the composition of basic rotations about $(n-2)$-dimensional axes will always again be a rotation about an $(n-2)$-dimensional axis. In these higher dimensions, general (compositions of) rotations cannot be so simply described. Such a (generalized) rotation may have an 'axis' (i.e. a space that is left undisturbed by the rotational motion) whose dimension can take a variety of different values. Thus, for a Clifford algebra in n dimensions, we need a hierarchy of different kinds of entity to represent such different kinds of rotation. In fact, it turns out to be better to start with something that is even more elementary than a rotation through π, namely a *reflection* in an $(n-1)$-dimensional (hyper)plane. A composition of two such reflections (with respect to two such planes that are perpendicular) provides a rotation through π, giving these previously basic π-rotations as 'secondary' entities, the primary entities being the reflections.[11.6]

We label these basic reflections $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_n$, where γ_r reverses the rth coordinate axis, while leaving all the others alone. For the appropriate type of 'spinorial object', reflecting it twice in the same direction gives the negative of the object, so we have n quaternion-like relations,

$$\gamma_1^2 = -1, \quad \gamma_2^2 = -1, \quad \gamma_3^2 = -1, \quad \ldots, \quad \gamma_n^2 = -1,$$

satisfied by these primary reflections. The secondary entities, representing our original π-rotations, are products of pairs of distinct γ's, and these products have anticommutation properties (rather like quaternions):

$$\gamma_p\gamma_q = -\gamma_q\gamma_p \quad (p \neq q).$$

In the particular case of three dimensions ($n = 3$), we can define the three different 'second-order' quantities

$$\mathbf{i} = \gamma_2\gamma_3, \quad \mathbf{j} = \gamma_3\gamma_1, \quad \mathbf{k} = \gamma_1\gamma_2,$$

[11.6] Find the geometrical nature of the transformation, in Euclidean 3-space, which is the composition of two reflections in planes that are not perpendicular.

and it is readily checked that these three quantities **i**, **j**, and **k** satisfy the quaternion algebra laws (Hamilton's 'Brougham Bridge' equations).[11.7]

The general element of the Clifford algebra for an n-dimensional space is a sum of real-number multiples (i.e. a linear combination) of products of sets of distinct γ's. The first-order ('primary') entities are the n different individual quantities γ_p. The second-order ('secondary') entities are the $\frac{1}{2}n(n-1)$ independent products $\gamma_p\gamma_q$ (with $p<q$); there are $\frac{1}{6}n(n-1)(n-2)$ independent third-order entities $\gamma_p\gamma_q\gamma_r$ (with $p<q<r$), $\frac{1}{24}n(n-1)(n-2)(n-3)$ independent fourth-order entities, etc., and finally the single nth-order entity $\gamma_1\gamma_2\gamma_3\cdots\gamma_n$. Taking all these, together with the single zeroth-order entity 1, we get

$$1+n+\frac{1}{2}n(n-1)+\frac{1}{6}n(n-1)(n-2)+\cdots+1=2^n$$

entities in all,[11.8] and the general element of the Clifford algebra is a linear combination of these. Thus the elements of a Clifford algebra constitute a 2^n-dimensional algebra over the reals, in the sense described in §11.1. They form a ring with identity but, unlike quaternions, they do not form a division ring.

One reason that Clifford algebras are important is for their role in defining spinors. In physics, spinors made their appearance in Dirac's famous equation for the electron (Dirac 1928), the electron's state being a spinor quantity (see Chapter 24). A spinor may be thought of as an object upon which the elements of the Clifford algebra act as operators, such as with the basic reflections and rotations of a 'spinorial object' that we have been considering. The very notion of a 'spinorial object' is somewhat confusing and non-intuitive, and some people prefer to resort to a purely (Clifford-) algebraic[11] approach to their study. This certainly has its advantages, especially for a general and rigorous n-dimensional discussion; but I feel that it is important also not to lose sight of the geometry, and I have tried to emphasize this aspect of things here.

In n dimensions,[12] the full space of spinors (sometimes called *spin-space*) is $2^{n/2}$-dimensional if n is even, and $2^{(n-1)/2}$-dimensional if n is odd. When n is even, the space of spinors splits into two independent spaces (sometimes called the spaces of 'reduced spinors' or 'half-spinors'), each of which is $2^{(n-2)/2}$-dimensional; that is, each element of the full space is the sum of two elements—one from each of the two reduced spaces. A reflection in the (even) n-dimensional space converts one of these reduced spin-spaces into the other. The elements of one reduced spin-space have a certain 'chirality' or 'handedness'; those of the other have the opposite chirality. This appears to have deep importance in physics, where I here refer to the spinors for ordinary 4-dimensional spacetime. The two reduced spin-spaces are each 2-dimensional, one referring to right-handed entities and the other to left-handed ones. It seems that Nature assigns a different role to each of these two reduced spin-spaces, and it is through this fact that physical processes that are reflection non-invariant can emerge. It was, indeed, one of the most striking (and some would say 'shocking') unprecedented discoveries of 20th-century physics (theoretically predicted by Chen Ning Yang and Tsung Dao Lee, and experimentally confirmed by Chien-Shiung Wu and her group, in 1957) that there are actually fundamental processes in Nature which do not occur in their mirror-reflected form. I shall be returning to these foundational issues later (§§25.3,4, §32.2, §§33.4,7,11,14).

Spinors also have an important technical mathematical value in various different contexts[13] (see §§22.8–11, §§23.4,5, §§24.6,7, §§32.3,4, §§33.4,6,8,11), and they can be of practical use in certain types of computation. Because of the 'exponential' relation between the dimension of the spin-space ($2^{n/2}$, etc.) and the dimension n of the original space, it is not surprising that spinors are better practical tools when n is reasonably small. For ordinary 4-dimensional spacetime, for example, each reduced spin-space has dimension only 2, whereas for modern 11-dimensional 'M-theory' (see §31.14), the spin-space has 32 dimensions.

[11.7] Show this.

[11.8] Explain all this counting. [illegible]

11.6 Grassmann algebras

Finally, let me turn to Grassmann algebra. From the point of view of the above discussion, we may think of Grassmann algebra as a kind of degenerate case of Clifford algebra, where we have basic anticommuting generating elements $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3, \ldots, \boldsymbol{\eta}_n$, similar to the $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_n$ of the Clifford algebra, but where each η_s squares to *zero*, rather than to the -1 that we have in the Clifford case:

$$\boldsymbol{\eta}_1^2=0, \quad \boldsymbol{\eta}_2^2=0, \quad \ldots, \quad \boldsymbol{\eta}_n^2=0.$$

The anticommutation law

$$\boldsymbol{\eta}_p\boldsymbol{\eta}_q=-\boldsymbol{\eta}_q\boldsymbol{\eta}_p$$

holds as before, except that the Grassmann algebra is now more 'systematic' than the Clifford algebra, because we do not have to specify '$p\neq q$' in this equation. The case $\boldsymbol{\eta}_p\boldsymbol{\eta}_p=-\boldsymbol{\eta}_p\boldsymbol{\eta}_p$ simply re-expresses $\boldsymbol{\eta}_p^2=0$.

Indeed, Grassmann algebras are more primitive and universal than Clifford algebras, as they depend only upon a minimal amount of local structure. Basically, the point is that the Clifford algebra needs to 'know' what 'perpendicular' means, so that ordinary rotations can be