CHAPTER 4

SOLUTION OF BOUNDARY-VALUE PROBLEMS

In the previous two chapters a number of elementary boundary-value problems were solved by means of certain specialized techniques such as the method of images, inversion in a sphere or cylinder, and the use of Gauss' law. These methods, when applicable, lead to the required solutions for the scalar potential Φ and the electric field in a very direct manner. However, for many problems that occur in practice, these special techniques are too restricted in their scope. In this chapter we shall examine a much more general approach, that is, the method of separation of variables. For this method it is necessary that the boundaries over which the potential or its normal derivative is specified coincide, at least piecewise, with the constant coordinate surfaces in a suitable orthogonal curvilinear coordinate system. Furthermore, it is necessary that the partial differential equation in question be separable in the appropriate system of coordinates so that the solution can be represented as a product of three functions which individually are a function of one coordinate variable only. When the solution can be represented in such a product form, the partial differential equation is said to be separable.

In three dimensions Laplace's equation is separable in 11 different coordinate systems. Among these systems are rectangular, spherical, and cylindrical coordinates, all of which will be examined in this chapter. In two dimensions Laplace's equation is separable in virtually an infinite number of different two-dimensional coordinate systems. We shall show that any coordinate system which is generated by an analytic function $W = F(Z)$, where Z is the complex variable $x + jy$, is a coordinate system in which Laplace's equation is separable.

Boundary-value problems are usually classified into three types. If the value of the potential is specified everywhere on the whole boundary, this boundary condition is referred to as a Dirichlet boundary condition. If, on the other hand, the normal derivative of Φ (this is proportional to the charge density) is specified on the whole boundary, we refer to this as Neumann boundary conditions. Finally, if the potential is specified on part of the boundary and the normal derivative $\partial\Phi/\partial n$ on the remainder, we have a mixed boundary-value problem. The method of solution in all three cases is essentially the same. It is not possible to specify both Φ and its normal derivative in an arbitrary manner over a common portion of the boundary since this overspecifies the problem. For example, if Φ is specified on the whole boundary, then since the solution to Laplace's equation is unique, $\partial\Phi/\partial n$ is completely determined. Thus, in this case, there is no choice in the values that $\partial\Phi/\partial n$ can have on the boundary. Similar remarks also apply to the Neumann and mixed boundary-value problems.

As was demonstrated in Sec. 2.9, if we can find a solution to Laplace's equation that satisfies all the required boundary conditions, then this solution is unique. Similar uniqueness theorems are readily established for other types of partial differential equations also. Likewise, the method of separation of variables, as discussed in this chapter in connection with Laplace's equation, is applicable to all partial differential equations.

If the boundaries do not coincide with constant coordinate surfaces in a coordinate system for which the partial differential equation is separable, then the method of separation of variables is of little use. In these cases (which do occur very often in practice) approximate methods of analysis or experimental techniques must be used.

4.1. Rectangular Coordinates

In rectangular coordinates Laplace's equation for the scalar potential Φ is

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{4.1}$$

To determine if this equation is separable in rectangular coordinates, we assume a product solution for Φ of the form

$$\Phi = f(x)g(y)h(z) \tag{4.2}$$

where f, g, and h are functions of x, y, and z, respectively, only. Substituting into (4.1) gives

$$ghf'' + fhg'' + fgh'' = 0$$

where $f'' = d^2f/dx^2$, $g'' = d^2g/dy^2$, and $h'' = d^2h/dz^2$. Dividing by fgh gives

$$\frac{f''}{f} + \frac{g''}{g} + \frac{h''}{h} = 0 \tag{4.3}$$

Each term is a function of one variable only; for example, f''/f depends on x only. If we keep y and z constant and vary x only, the term f''/f can possibly vary. However, the sum of the three terms must equal zero, and therefore each term must be equal to a constant in order for (4.3) to hold for all arbitrary values of x, y, and z. Thus we must have

$$\frac{d^2f}{dx^2} + k_x^2 f = 0 \tag{4.4a}$$

$$\frac{d^2g}{dy^2} + k_y^2 g = 0 \tag{4.4b}$$

$$\frac{d^2h}{dz^2} + k_z^2 h = 0 \tag{4.4c}$$

where k_x, k_y, and k_z are constants (as yet arbitrary) called separation constants. The only restriction on the separation constants so far is that the sum be equal to zero, so that (4.3) will hold; i.e.,

$$k_x^2 + k_y^2 + k_z^2 = 0 \tag{4.5}$$

By means of the above procedure we have reduced the solution of the partial differential equation to that of solving three ordinary differential equations. It is this property that leads to the designation "method of separation of variables" for the technique used. In practice, the separation constants are determined by the boundary conditions which the potential or its normal derivative must satisfy. The details will become clear from the examples to be discussed, but first we shall consider some special cases.

Case 1

If $k_x = 0$ but k_y and k_z are not zero, then from (4.5) it is seen that $k_z = \pm jk_y$. When $k_x = 0$, the solution to (4.4*a*) is $f = A_1x + A_2$, where A_1 and A_2 are arbitrary constants. The solution to (4.4*b*), with k_y^2 chosen to be positive, is then

$$g = B_1 \sin k_y y + B_2 \cos k_y y$$

where B_1 and B_2 are arbitrary constants. This solution is readily verified by substitution in (4.4*b*). Since $k_z^2 = -k_y^2$, (4.4*c*) becomes

$$\frac{d^2h}{dz^2} - k_y^2 h = 0$$

with a general solution of the form

$$h = C_1 \sinh k_y z + C_2 \cosh k_y z$$

where C_1 and C_2 are constants. In the above solutions we could equally well choose the exponential forms

$$g = B_1 e^{jk_y y} + B_2 e^{-jk_y y}$$
$$h = C_1 e^{k_y z} + C_2 e^{-k_y z}$$

According to the mathematical theory, any two independent solutions constitute a general solution for a second-order differential equation. In the present case the solution for Φ is

$$\Phi = fgh = (A_1x + A_2)(B_1 \sin k_y y + B_2 \cos k_y y)(C_1 \sinh k_y z + C_2 \cosh k_y z) \tag{4.6}$$

The constants in this solution would normally be determined by the boundary conditions. If k_y^2 had been chosen as a negative constant, the hyperbolic and trigonometric functions in (4.6) would be interchanged.

Case 2

If $k_x = k_y = 0$, it is necessary that $k_z = 0$ also in order for (4.5) to hold. In this case the solution is of the form

$$\Phi = (A_1x + A_2)(B_1y + B_2)(C_1z + C_2) \tag{4.7}$$

Case 3

If k_x and k_y are both positive real constants, then from (4.5) we have

$$k_z = \pm j(k_x^2 + k_y^2)^{1/2} \tag{4.8}$$

In this case the general solution for Φ is

$$\Phi = (A_1 \sin k_x x + A_2 \cos k_x x)(B_1 \sin k_y y + B_2 \cos k_y y)(C_1 \sinh |k_z|z + C_2 \cosh |k_z|z) \tag{4.9}$$

where $|k_z| = (k_x^2 + k_y^2)^{1/2}$. Other variations of (4.9) can be derived by cyclic permutation of the variables.

In many problems it is found that a combination of the various solutions discussed above must be used in order to satisfy all the required boundary conditions. Also, it is usually found that the separation constants can take on an infinite sequence of values. The general solution for Φ is then given by a summation over all the possible individual solutions. This particular property makes the general solution extremely flexible in that it can be made to satisfy any arbitrary boundary condition. These points will be further elaborated in the course of solutions of the following examples.

Example 4.1. A Rectangular Dirichlet Boundary-value Problem. We wish to find a solution to Laplace's equation in the interior of a rectangular enclosure such that the potential Φ reduces to zero on all sides except the side at $z = c$. On the surface $z = c$, $0 \le x \le a$, $0 \le y \le b$, the potential Φ is equal to the specified value $V(x,y)$. With reference to Fig. 4.1, it should be noted that the boundaries coincide with constant coordinate surfaces.

The boundary condition $\Phi = 0$ on the two faces $x = 0, a$ must be satisfied for all values of y and z on these faces. This condition is met if $f(x)$

vanishes at $x = 0, a$. Since the solution for f is of the form

$$f = A_1 x + A_2 \qquad k_x = 0$$
$$f = A_1 \sin k_x x + A_2 \cos k_x x \qquad k_x \neq 0$$

we see that the first solution for $k_x = 0$ is valid only if A_1 and A_2 are zero.

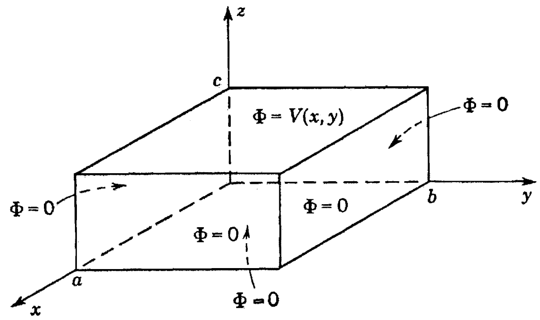

FIG. 4.1. A rectangular parallelepiped with specified boundary conditions.

This is a trivial solution; so the second form must be chosen. For f to equal zero at $x = 0$, we must choose $A_2 = 0$. Hence we have

$$f = A_1 \sin k_x x$$

Now f must equal zero at $x = a$ also, and hence $\sin k_x a = 0$. From this result we see that the separation constant k_x is given by

$$k_x = \frac{n\pi}{a} \qquad n = 1, 2, 3, \ldots$$

The function $\sin (n\pi x/a)$ is called an eigenfunction (proper function) and $n\pi/a$ an eigenvalue. The most general solution for f is

$$f = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{a} \tag{4.10a}$$

where the A_n are as yet arbitrary constants.

What we have said about the function $f(x)$ is applicable to the function $g(y)$ also. It is therefore not difficult to see that a general solution for $g(y)$ that vanishes at $y = 0, b$ is

$$g(y) = \sum_{m=1}^{\infty} B_m \sin \frac{m\pi y}{b} \qquad m = 1, 2, 3, \ldots \tag{4.10b}$$

For the nmth solution for fg we have $k_x^2 = (n\pi/a)^2$ and $k_y^2 = (m\pi/b)^2$.

The corresponding value of k_z must be

$$k_z = \pm j\left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right]^{1/2} = \pm j\Gamma_{nm}$$

Of the two possible solutions $\sinh |k_z|z$ and $\cosh |k_z|z$ for $h(z)$, the hyperbolic sine function must be chosen, since this is the only solution that vanishes at $z = 0$. The general solution for Φ is thus of the form

$$\Phi = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_nB_m \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\sinh\left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right]^{1/2} z \quad (4.11)$$

We note that the nature of the boundary conditions has led to a solution such as was discussed under case 3.

In order to determine the coefficients A_n and B_m we must impose the final boundary condition at $z = c$. From (4.11) we obtain

$$V(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b} \quad (4.12)$$

where for convenience we have defined

$$\begin{aligned} C_{nm} &= A_nB_m \sinh\left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right]^{1/2} c \\ &= A_nB_n \sinh \Gamma_{nm}\, c \end{aligned}$$

The eigenfunctions $\sin (n\pi x/a)$ and $\sin (m\pi y/b)$ occurring in (4.12) have an orthogonality property that enables C_{nm} to be determined. This orthogonality property is the vanishing of the following integrals:

$$\int_0^a \sin\frac{n\pi x}{a}\sin\frac{s\pi x}{a}\,dx = 0 \qquad n \neq s \quad (4.13a)$$

$$\int_0^b \sin\frac{m\pi y}{b}\sin\frac{s\pi y}{b}\,dy = 0 \qquad m \neq s \quad (4.13b)$$

When $n = s$ or $m = s$ we have

$$\int_0^a \sin^2\frac{n\pi x}{a}\,dx = \frac{a}{2} \quad (4.14a)$$

$$\int_0^b \sin^2\frac{m\pi y}{b}\,dy = \frac{b}{2} \quad (4.14b)$$

Orthogonality properties similar to these are found to apply to the eigenfunctions that occur in other coordinate systems as well.

The above orthogonality properties (4.13) are readily proved by direct integration. However, since we shall be dealing with more complicated functions later on, it will be instructive to prove these properties by a more general method. Let $f_n = \sin (n\pi x/a)$ and $f_s = \sin (s\pi x/a)$. Multiplying the equation satisfied by f_n by f_s and similarly the equation for f_s by f_n gives

$$f_s \frac{d^2 f_n}{dx^2} + \left(\frac{n\pi}{a}\right)^2 f_n f_s = 0$$

$$f_n \frac{d^2 f_s}{dx^2} + \left(\frac{s\pi}{a}\right)^2 f_n f_s = 0$$

Subtracting the two equations and integrating over $0 \leq x \leq a$ gives

$$\left[\left(\frac{n\pi}{a}\right)^2 - \left(\frac{s\pi}{a}\right)^2\right] \int_0^a f_n f_s \, dx = \int_0^a \left(f_n \frac{d^2 f_s}{dx^2} - f_s \frac{d^2 f_n}{dx^2}\right) dx$$

If we integrate the right-hand side by parts once, we obtain

$$\int_0^a \left(f_n \frac{d^2 f_s}{dx^2} - f_s \frac{d^2 f_n}{dx^2}\right) dx = \left(f_n \frac{df_s}{dx} - f_s \frac{df_n}{dx}\right)\Bigg|_0^a - \int_0^a \left(\frac{df_n}{dx}\frac{df_s}{dx} - \frac{df_s}{dx}\frac{df_n}{dx}\right) dx$$

Since both f_n and f_s vanish at $x = 0$, a and the integrand in the integral on the right-hand side vanishes, it follows that the integral on the left-hand side is equal to zero. Hence, since $n \neq s$, it follows that

$$\int_0^a f_n f_s \, dx = 0 \qquad n \neq s$$

In a similar way (4.13*b*) may be proved.

Returning to (4.12) and multiplying both sides by $\sin (s\pi x/a)$ and $\sin (t\pi y/b)$, where s and t are integers, we obtain

$$\int_0^a \int_0^b V(x,y) \sin \frac{s\pi x}{a} \sin \frac{t\pi y}{b} \, dx \, dy = \frac{ab}{4} C_{st} \tag{4.15}$$

by virtue of (4.13) and (4.14). Equation (4.12) represents a double Fourier series for $V(x,y)$, and by virtue of the orthogonal properties of the eigenfunctions an equation for C_{st}, that is, (4.15), is readily obtained. If $V(x,y)$ is known, (4.15) can be evaluated.

Let us choose for $V(x,y)$ the form

$$V(x,y) = V_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$$

In this case all $C_{st} = 0$ except C_{11}, which from (4.15) has the value V_0. Thus the solution for Φ is

$$\begin{aligned} \Phi &= C_{11} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \frac{\sinh \Gamma_{11} z}{\sinh \Gamma_{11} c} \\ &= \frac{V_0}{\sinh \Gamma_{11} c} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \sinh \Gamma_{11} z \end{aligned} \tag{4.16}$$

where $$\Gamma_{11}^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$

If $V(x,y)$ was of the form

$$V(x,y) = V_0 \sin \frac{\pi y}{b}$$

instead, then all C_{st} for $t \neq 1$ are zero. From (4.15) the values of C_{s1} are

$$\begin{aligned}
\frac{ab}{4} C_{s1} &= \int_0^a \int_0^b V_0 \sin \frac{s\pi x}{a} \sin^2 \frac{\pi y}{b}\, dx\, dy \\
&= \frac{V_0 b}{2} \int_0^a \sin \frac{s\pi x}{a}\, dx \\
&= -\frac{V_0 b}{2} \frac{a}{s\pi} \cos \frac{s\pi x}{a} \Big|_0^a = \frac{V_0 b}{2} \frac{a}{s\pi} (1 - \cos s\pi) \\
&= \frac{V_0 ba}{s\pi} \qquad s = 1, 3, 5, \ldots
\end{aligned}$$

Hence $C_{s1} = 4V_0/s\pi$, and the solution for Φ becomes

$$\Phi = \sum_{s=1,3,5,\ldots}^{\infty} \frac{4V_0}{s\pi} \sin \frac{s\pi x}{a} \sin \frac{\pi y}{b} \frac{\sinh \Gamma_{s1} z}{\sinh \Gamma_{s1} c} \tag{4.17}$$

If the potential Φ is specified different from zero on some of the other faces as well, the complete solution to the problem is readily constructed by finding a solution for Φ similar to the above that vanishes on all sides but one. On the latter side, Φ is made to satisfy the required boundary condition. A superposition of these potential functions will then satisfy all the boundary conditions. For example, if the side $z = c$ is kept at a potential $V_1(x,y)$, side $x = a$ at a potential $V_2(y,z)$, and the side $y = b$ at a potential $V_3(x,z)$, we construct three potential functions Φ_1, Φ_2, and Φ_3 with the following properties. All Φ_i ($i = 1, 2, 3$) satisfy Laplace's equation. In addition, the Φ_i are determined so that the following boundary conditions are satisfied:

$$\begin{aligned}
\Phi_1 &= 0 && \text{on all sides except } z = c \\
\Phi_1 &= V_1(x,y) && \text{at } z = c \\
\Phi_2 &= 0 && \text{on all sides except } x = a \\
\Phi_2 &= V_2(y,z) && \text{at } x = a \\
\Phi_3 &= 0 && \text{on all sides except } y = b \\
\Phi_3 &= V_3(x,z) && \text{at } y = b
\end{aligned}$$

The potential $\Phi = \Phi_1 + \Phi_2 + \Phi_3$ is then a solution of Laplace's equation and in addition satisfies all the required boundary conditions. The solution for each Φ_i is similar to that used to obtain the solutions (4.16) and (4.17).

Example 4.2. A Two-dimensional Problem. As a second example consider a two-dimensional region with boundaries at $x = 0$, a, $y = 0$, b, as in Fig. 4.2. Let the boundary conditions be

$$\frac{\partial \Phi}{\partial x} = 0 \qquad x = 0 \tag{4.18a}$$
$$\Phi = 0 \qquad y = 0, b \tag{4.18b}$$
$$\Phi = V(y) \qquad x = a \tag{4.18c}$$

The potential Φ is independent of z (the structure is of infinite extent in the z direction), and hence is a solution of the two-dimensional Laplace equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \tag{4.19}$$

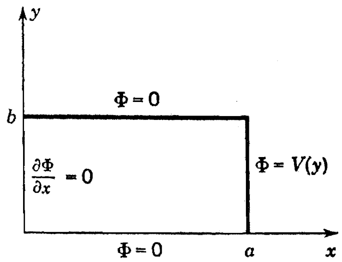

FIG. 4.2. A two-dimensional boundary-value problem.

Two-dimensional structures of the above form usually do not occur in practice; i.e., we do not have structures of infinite extent. However, we could in the present case be dealing equally well with a current flow problem where the current is confined to flow in a very thin sheet of conducting material. If we had a sheet of resistance paper and painted silver strips along the three sides $x = a$, $y = 0$, b, as in Fig. 4.3, and kept the side at $x = a$ at a constant potential V_0, we should have a two-dimensional problem similar to that illustrated in Fig. 4.2. The current density $\mathbf{J}$ is equal to the conductivity σ of the resistance sheet times the electric field and hence given by $\mathbf{J} = -\sigma \nabla \Phi$. The current flow lines would be similar to those sketched in Fig. 4.3. Since the sheet terminates along $x = 0$, the

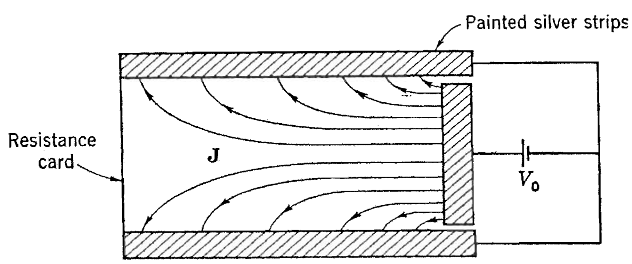

FIG. 4.3. A two-dimensional current flow problem.

current cannot flow normal to this edge and the boundary condition $\partial \Phi / \partial x = 0$ is satisfied.

To solve (4.19) subject to the boundary conditions (4.18), we assume a product solution $f(x)g(y)$. As before, we find that f and g satisfy (4.4a) and (4.4b), respectively. In addition, since $k_z = 0$, we must have $k_x = jk_y$, so that $k_x^2 + k_y^2 = 0$. In order to satisfy the boundary conditions at $y = 0$, b, each possible solution for $g(y)$ must be chosen as $\sin(n\pi y/b)$, where n is an integer. The corresponding solution for f must be either $\cosh(n\pi x/b)$ or $\sinh(n\pi x/b)$. Only the $\cosh(n\pi x/b)$ solution satisfies the boundary condition (4.18a) and hence is the solution chosen. The most general solution for Φ that satisfies the boundary conditions at $x = 0$, $y = 0$, and $y = b$ is thus

$$\Phi = \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi x}{b} \sin \frac{n\pi y}{b} \tag{4.20}$$

At $x = a$ we must have

$$\Phi = V(y) = \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi a}{b} \sin \frac{n\pi y}{b} \tag{4.21}$$

To determine the coefficients A_n we use the orthogonality property (4.13b) and also the result (4.14b) to expand $V(y)$ into a Fourier series. Multiplying both sides of (4.21) by $\sin(m\pi y/b)$ and integrating over y gives

$$A_m \frac{b}{2} \cosh \frac{m\pi a}{b} = \int_0^b V(y) \sin \frac{m\pi y}{b}\, dy \tag{4.22}$$

This equation determines A_m and hence completely specifies the potential function Φ.

If $V(y)$ is equal to a constant V_0, we have

$$\begin{aligned} A_m &= \frac{2V_0}{b \cosh(m\pi a/b)} \frac{b}{m\pi} (1 - \cos m\pi) \\ &= \frac{4V_0}{m\pi \cosh(m\pi a/b)} \qquad m = 1, 3, 5, \ldots \end{aligned} \tag{4.23}$$

Thus
$$\Phi = \frac{4V_0}{\pi} \sum_{m=1,3,\ldots}^{\infty} \frac{1}{m} \frac{\cosh(m\pi x/b)}{\cosh(m\pi a/b)} \sin \frac{m\pi y}{b} \tag{4.24}$$

This is the solution for the potential in the current flow problem illustrated in Fig. 4.3. The current density $\mathbf{J}$ is given by $-\sigma \nabla\Phi$.

If we modify the boundary conditions (4.18) and require that the side $y = b$ be kept at a constant potential V_1, we may satisfy this boundary condition by means of a partial solution $\Phi_1 = V_1 y/b$ corresponding to a choice $k_x = k_y = 0$ for the separation constants. For Φ we now choose

$$\Phi = V_1 \frac{y}{b} + \sum_{n=1}^{\infty} B_n \cosh \frac{n\pi x}{b} \sin \frac{n\pi y}{b} \tag{4.25}$$

At $x = 0$, clearly, $\partial\Phi/\partial x = 0$, since $V_1 y/b$ is not a function of x. Also at $y = 0$, $\Phi = 0$, while at $y = b$, $\Phi = V_1$. At $x = a$ we must have $\Phi = V(y)$, and hence

$$V(y) = V_1 \frac{y}{b} + \sum_{n=1}^{\infty} B_n \cosh \frac{n\pi a}{b} \sin \frac{n\pi y}{b} \tag{4.26}$$

In this case the coefficients B_n are given by

$$B_n = \frac{2}{b}\left(\cosh \frac{n\pi a}{b}\right)^{-1} \int_0^b \left[-V_1 \frac{y}{b} + V(y)\right] \sin \frac{n\pi y}{b}\, dy \tag{4.27}$$

As a third modification consider the same rectangular region illustrated in Fig. 4.2 but with the following boundary conditions:

$$\begin{aligned} \frac{\partial\Phi}{\partial x} &= 0 & x &= 0 \\ \Phi &= 0 & y &= 0 \\ \Phi &= V(y) & x &= a \\ \frac{\partial\Phi}{\partial y} &= p(x) & y &= b \end{aligned}$$

The function $p(x)$ is equal to $\rho(x)/\epsilon_0$, where ρ is the charge density on the side $y = b$. To solve this problem we construct two partial solutions Φ_1 and Φ_2 with the following properties:

For Φ_1

$$\begin{aligned} \frac{\partial\Phi_1}{\partial x} &= 0 & x &= 0 \\ \Phi_1 &= 0 & y &= 0 \\ \frac{\partial\Phi_1}{\partial y} &= 0 & y &= b \\ \Phi_1 &= V(y) & x &= a \end{aligned}$$

For Φ_2

$$\begin{aligned} \frac{\partial\Phi_2}{\partial x} &= 0 & x &= 0 \\ \Phi_2 &= 0 & y &= 0 \\ \Phi_2 &= 0 & x &= a \\ \frac{\partial\Phi_2}{\partial y} &= p(x) & y &= b \end{aligned}$$

A superposition of Φ_1 and Φ_2 gives a potential $\Phi = \Phi_1 + \Phi_2$, which satisfies all the boundary conditions. The reader may readily verify that appropriate solutions for Φ_1 and Φ_2 are

$$\Phi_1 = \sum_{n=1}^{\infty} A_n \cosh\left(\frac{2n-1}{2b}\pi x\right) \sin\left(\frac{2n-1}{2b}\pi y\right) \tag{4.28a}$$

where $$A_n \frac{b}{2} \cosh\left(\frac{2n-1}{2b}\pi a\right) = \int_0^b V(y) \sin\left(\frac{2n-1}{2b}\pi y\right) dy$$

and $$\Phi_2 = \sum_{m=1}^{\infty} B_m \cos\left(\frac{2m-1}{2a}\pi x\right) \sinh\left(\frac{2m-1}{2a}\pi y\right) \tag{4.28b}$$

where

$$B_m \frac{2m-1}{2a}\pi \frac{a}{2} \cosh\left(\frac{2m-1}{2a}\pi b\right) = \int_0^a p(x) \cos\left(\frac{2m-1}{2a}\pi x\right) dx$$

In (4.28) the functions

$$\sin\frac{2n-1}{2b}\pi y \qquad \text{and} \qquad \cos\frac{2m-1}{2a}\pi x$$

are orthogonal over the respective ranges $0 \leq y \leq b$ and $0 \leq x \leq a$, and hence the usual Fourier series analysis could be used to determine the coefficients A_n and B_m.

The above technique of superimposing partial solutions in order to satisfy arbitrary boundary conditions on a number of sides is also applicable to problems occurring in other coordinate systems as well.

4.2. Cylindrical Coordinates

In a cylindrical coordinate system r, ϕ, z, Laplace's equation is

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{4.29}$$

This equation is separable; so solutions of the form $\Phi = f(r)g(\phi)h(z)$ exist. Substituting into (4.29) gives

$$gh\frac{1}{r}\frac{d}{dr} r \frac{df}{dr} + \frac{fh}{r^2}\frac{d^2 g}{d\phi^2} + fg\frac{d^2 h}{dz^2} = 0$$

Dividing by fgh and multiplying by r^2 gives

$$\left(\frac{r}{f}\frac{d}{dr} r \frac{df}{dr}\right) + \left(\frac{1}{g}\frac{d^2 g}{d\phi^2}\right) + r^2\left(\frac{1}{h}\frac{d^2 h}{dz^2}\right) = 0 \tag{4.30}$$

The second term is a function of ϕ only, and hence (4.30) can hold for all values of r, ϕ, and z only if this term is constant. Thus we must have

$$\frac{d^2 g}{d\phi^2} + \nu^2 g = 0 \tag{4.31}$$

where ν^2 is a separation constant. In many practical problems the whole range $0 \le \phi \le 2\pi$ is involved, and since Φ must be single-valued, that is, $\Phi(2\pi) = \Phi(0)$, ν must be equal to an integer n. The solutions to (4.31) are clearly

$$g = B_1 \sin n\phi + B_2 \cos n\phi \tag{4.32a}$$

or

$$g = B_1 e^{jn\phi} + B_2 e^{-jn\phi} \tag{4.32b}$$

when $\nu = n$. Of course, (4.32) is still the appropriate solution even if n is not an integer.

The second term in (4.30) may now be replaced by $-\nu^2$. Making this substitution and dividing by r^2 gives

$$\left(\frac{1}{rf}\frac{d}{dr}r\frac{df}{dr} - \frac{\nu^2}{r^2}\right) + \frac{1}{h}\frac{d^2h}{dz^2} = 0 \tag{4.33}$$

Each term in this equation is a function of one variable only and must equal a constant if the equation is to hold for all values of r and z. Consequently, we have

$$\frac{d^2h}{dz^2} + k_z{}^2 h = 0 \tag{4.34}$$

$$\frac{1}{r}\frac{d}{dr}r\frac{df}{dr} - \left(\frac{\nu^2}{r^2} + k_z{}^2\right) f = 0 \tag{4.35}$$

Equation (4.34) is of the type already considered and has solutions of the form

$$h = C_1 \sin k_z z + C_2 \cos k_z z \tag{4.36a}$$

or if $k_z = j\Gamma$ and Γ is real,

$$h = C_1 \sinh \Gamma z + C_2 \cosh \Gamma z \tag{4.36b}$$

Equation (4.35) is Bessel's equation, and the two independent solutions are called Bessel's functions of the first and second kinds and of order ν. In the special case when $k_z = 0$, the solution reduces to a simple power of r. We shall consider this special case first.

Solution When $k_z = 0$

When the potential has no variation with z, the separation constant $k_z = 0$ and (4.35) becomes

$$\frac{1}{r}\frac{d}{dr}r\frac{df}{dr} - \frac{\nu^2}{r^2}f = 0 \tag{4.37}$$

Let us see if a simple function such as $f = r^\alpha$ will be a solution. Substituting into (4.37) gives

$$\frac{1}{r}\frac{d}{dr}r\frac{dr^\alpha}{dr} - \nu^2 r^{\alpha-2} = (\alpha^2 - \nu^2)r^{\alpha-2} = 0$$

Hence r^α is a possible solution provided $\alpha = \pm\nu$. A general solution for Φ in this case is (for $\nu = 0 \quad f = A_0 \ln r$)

$$\Phi = \sum_{n=1}^{\infty} [r^n(A_n \sin n\phi + B_n \cos n\phi) + A_0 \ln r + r^{-n}(C_n \sin n\phi + D_n \cos n\phi)] \quad (4.38)$$

where we have chosen $\nu = n$, and A_n, B_n, C_n, D_n are amplitude constants.

Example 4.3. Dielectric Cylinder in a Uniform Applied Field. Consider a dielectric cylinder of radius r_0 and permittivity ϵ infinitely long and parallel to the z axis and placed in a uniform electrostatic field $\mathbf{E}_0$ directed along the x axis, as in Fig. 4.4. We wish to determine the induced potential and field for all values of r and ϕ.

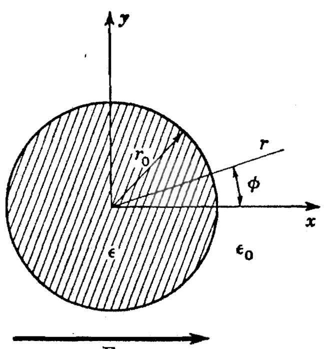

FIG. 4.4. A dielectric cylinder in a uniform applied field $\mathbf{E}_0$.

In cylindrical coordinates, $x = r\cos\phi$, and hence $\mathbf{E}_0$ may be considered as the field arising from an applied potential Φ_0 given by

$$\Phi_0 = -E_0 r \cos\phi \quad (4.39)$$

since $-\nabla\Phi_0 = \mathbf{E}_0$. Let Φ be the induced potential. Since Φ_0 varies with ϕ according to $\cos\phi$, the induced potential Φ will also. This may be seen by noting that the boundary conditions at $r = r_0$ must hold for all values of ϕ, and since $\cos\phi$ is orthogonal to $\cos n\phi$ and $\sin n\phi$ for $n \neq 1$, only the $n = 1$ term in the general solution (4.38) is coupled to the applied potential. The potential Φ must be finite at $r = 0$ and vanish as r approaches infinity. Hence a suitable form for Φ is

$$\Phi = \begin{cases} Ar\cos\phi & r \le r_0 \\ Br^{-1}\cos\phi & r \ge r_0 \end{cases}$$

At $r = r_0$, the total potential must be continuous across the boundary, so that

$$Ar_0\cos\phi + \Phi_0(r_0) = Br_0^{-1}\cos\phi + \Phi_0(r_0)$$

or

$$B = r_0^2 A$$

Also at $r = r_0$ the radial component of the displacement flux density, that is, ϵE_r, must be continuous. Thus

$$-\epsilon\frac{\partial}{\partial r}(Ar\cos\phi - E_0 r\cos\phi) = -\epsilon_0\frac{\partial}{\partial r}(Br^{-1}\cos\phi - E_0 r\cos\phi)$$

or

$$\epsilon(A - E_0) = -\epsilon_0\left(\frac{B}{r_0^2} + E_0\right)$$

The solutions for A and B are now readily found and are

$$A = \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} E_0 \tag{4.40a}$$

$$B = r_0^2 A \tag{4.40b}$$

In the interior of the cylinder the total potential is

$$\Phi + \Phi_0 = -\frac{2\epsilon_0}{\epsilon + \epsilon_0} E_0 r \cos \phi \tag{4.41}$$

The field is still uniform but smaller in magnitude than the applied field E_0. This reduction in the internal field is produced by the depolarizing field set up by the equivalent dipole polarization charge on the surface of the cylinder. The total internal field is

$$E_i = \frac{2\epsilon_0}{\epsilon + \epsilon_0} E_0 \tag{4.42}$$

Outside the cylinder the induced field is $\mathbf{E}_e$, where

$$\mathbf{E}_e = \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} E_0 \left(\frac{r_0}{r}\right)^2 (\mathbf{a}_r \cos \phi + \mathbf{a}_\phi \sin \phi) \tag{4.43}$$

This field is identical with that produced by a line dipole located at the origin.

Solution When $k_z \neq 0$

When $k_z \neq 0$, the function $f(r)$ is a solution of Bessel's equation of order n for $\nu = n$:

$$\boxed{\frac{1}{r}\frac{d}{dr} r \frac{df}{dr} + \left(\Gamma^2 - \frac{\nu^2}{r^2}\right) f = 0} \tag{4.44}$$

nel nostro caso la Φ è costante rispetto l'angolo perciò v=n=0, e la soluzione è l'eq di BESSEl di ordine zero

where we have chosen $k_z = j\Gamma$. Substitution of a general power series in r into this equation shows that the two independent solutions are

$$J_n(\Gamma r) = \sum_{m=0}^{\infty} \frac{(-1)^m (\Gamma r/2)^{n+2m}}{m!(n+m)!} \tag{4.45a}$$

$$\begin{aligned} Y_n(\Gamma r) = {} & \frac{2}{\pi}\left(\gamma + \ln \frac{\Gamma r}{2}\right) J_n(\Gamma r) \\ & - \frac{1}{\pi} \sum_{m=0}^{n-1} \frac{(n-m-1)!}{m!} \left(\frac{2}{\Gamma r}\right)^{n-2m} \\ & - \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m (\Gamma r/2)^{n+2m}}{m!(n+m)!} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} + 1 \right. \\ & \left. + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+m}\right) \qquad \gamma = 0.5772 \end{aligned} \tag{4.45b}$$

The first series (4.45a) defines the Bessel function of the first kind and order n, while the second series defines the Bessel function of the second kind and order n. These functions are tabulated in many places. As seen from (4.45b), the function $Y_0(\Gamma r)$ has a logarithmic singularity at $r = 0$. For problems that include the origin, this function will not be

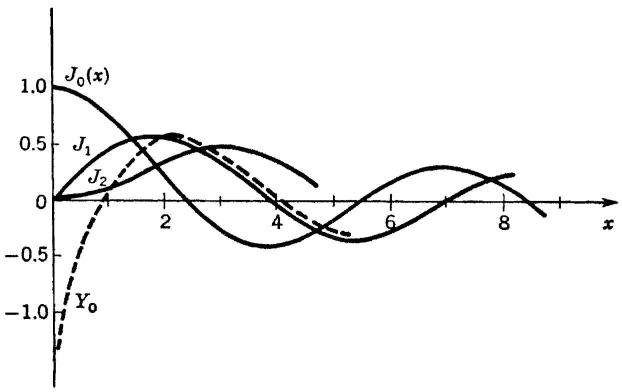

FIG. 4.5. A plot of a few of the lower-order Bessel functions.

part of the solution unless a line source is located at the origin. For $n > 0$, Y_n has a singularity of order r^{-n}. For large values of r, Bessel's functions reduce to damped sinusoids:

$$\lim_{r\to\infty} J_n(\Gamma r) = \sqrt{\frac{2}{\Gamma r\pi}} \cos\left(\Gamma r - \frac{\pi}{4} - \frac{n\pi}{2}\right) \tag{4.46a}$$

$$\lim_{r\to\infty} Y_n(\Gamma r) = \sqrt{\frac{2}{\Gamma r\pi}} \sin\left(\Gamma r - \frac{\pi}{4} - \frac{n\pi}{2}\right) \tag{4.46b}$$

A plot of a few of the lower-order Bessel functions is given in Fig. 4.5.

If k_z is real, then Γ is imaginary. In this case the two independent solutions to Bessel's equation are still given by the series (4.45a) and (4.45b). However, for convenience, new symbols have been adopted to represent Bessel functions of imaginary argument; that is, by definition,

$$I_n(x) = j^{-n}J_n(jx) = j^nJ_n(-jx) \tag{4.47a}$$

$$K_n(x) = \frac{\pi}{2} j^{n+1}[J_n(jx) + jY_n(jx)] \tag{4.47b}$$

The functions I_n and K_n are defined so that they are real when x is real. I_n is called the modified Bessel function of the first kind, while K_n is called the modified Bessel function of the second kind. In the definition of K_n, a linear combination of $J_n(jx)$ and $Y_n(jx)$ is chosen in order to make $K_n(x)$ a decaying exponential function for large values of x. When

x is very large, the asymptotic values of I_n and K_n are

$$\underset{x\to\infty}{I_n(x)} = \frac{e^x}{\sqrt{2\pi x}} \tag{4.48a}$$

$$\underset{x\to\infty}{K_n(x)} = \sqrt{\frac{\pi}{2x}}\, e^{-x} \tag{4.48b}$$

It should be noted that these functions do not ever equal zero and that only $K_n(x)$ is finite (vanishes) at infinity.

Some useful properties of the Bessel functions J_n and Y_n are given below. Although the formulas are written specifically for J_n, they apply without change if J_n is replaced by Y_n.

Differentiation Formulas

$$J_0'(\Gamma r) = \frac{dJ_0(\Gamma r)}{d(\Gamma r)} = -J_1(\Gamma r) \tag{4.49a}$$

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x) \tag{4.49b}$$

where x has been written for Γr.

Recurrence Formula

$$\frac{2n}{x} J_n(x) = J_{n+1}(x) + J_{n-1}(x) \tag{4.50}$$

If J_{n-1} and J_n are known, this formula permits J_{n+1} to be found.

Integrals

$$\int xJ_n(\alpha x)J_n(\beta x)\,dx = \frac{x}{\alpha^2-\beta^2}[\beta J_n(\alpha x)J_{n-1}(\beta x) - \alpha J_{n-1}(\alpha x)J_n(\beta x)] \qquad \alpha \neq \beta \tag{4.51a}$$

$$\int xJ_n^2(\alpha x)\,dx = \frac{x^2}{2}[J_n^2(\alpha x) - J_{n-1}(\alpha x)J_{n+1}(\alpha x)] = \frac{x^2}{2}\left[J_n'^2(\alpha x) + \left(1 - \frac{n^2}{\alpha^2x^2}\right)J_n^2(\alpha x)\right] \tag{4.51b}$$

Bessel functions have a useful orthogonality property that permits an arbitrary function $f(r)$ defined over an interval $0 \le r \le a$ to be expanded into a Fourier-type series. The function $f(r)$ must be at least piecewise continuous if the expansion is to be valid. Let Γ_{nm} $(m = 1, 2, 3, \ldots)$ be the sequence of values of Γ that makes $J_n(\Gamma a) = 0$; that is, $\Gamma_{nm}a$ is the mth root of $J_n(x) = 0$. For any two roots of $J_n(x) = 0$, say Γ_{nm} and Γ_{ns}, we have

$$J_n(\Gamma_{ns}r)\left[\frac{1}{r}\frac{d}{dr}r\frac{dJ_n(\Gamma_{nm}r)}{dr} + \left(\Gamma_{nm}^2 - \frac{n^2}{r^2}\right)J_n(\Gamma_{nm}r)\right] = 0$$

$$J_n(\Gamma_{nm}r)\left[\frac{1}{r}\frac{d}{dr}r\frac{dJ_n(\Gamma_{ns}r)}{dr} + \left(\Gamma_{ns}^2 - \frac{n^2}{r^2}\right)J_n(\Gamma_{ns}r)\right] = 0$$

Subtracting the two equations, multiplying by r, and integrating with respect to r over $0 \leq r \leq a$ gives

$$(\Gamma_{nm}^2 - \Gamma_{ns}^2) \int_0^a rJ_n(\Gamma_{nm}r)J_n(\Gamma_{ns}r)\, dr = \int_0^a \left[J_n(\Gamma_{nm}r) \frac{d}{dr} r \frac{dJ_n(\Gamma_{ns}r)}{dr} - J_n(\Gamma_{ns}r) \frac{d}{dr} r \frac{dJ_n(\Gamma_{nm}r)}{dr} \right] dr$$

Integrating the right-hand side by parts once gives

$$\begin{aligned}(\Gamma_{nm}^2 - \Gamma_{ns}^2) &\int_0^a rJ_n(\Gamma_{nm}r)J_n(\Gamma_{ns}r)\, dr \\ &= r\left[J_n(\Gamma_{nm}r) \frac{dJ_n(\Gamma_{ns}r)}{dr} - J_n(\Gamma_{ns}r) \frac{dJ_n(\Gamma_{nm}r)}{dr} \right]_0^a \\ &\quad - \int_0^a r \left[\frac{dJ_n(\Gamma_{ns}r)}{dr} \frac{dJ_n(\Gamma_{nm}r)}{dr} - \frac{dJ_n(\Gamma_{nm}r)}{dr} \frac{dJ_n(\Gamma_{ns}r)}{dr} \right] dr\end{aligned}$$

The integrand on the right-hand side vanishes. Likewise, the integrated terms vanish since $r = 0$ at the lower limit while at the upper limit $J_n(\Gamma_{nm}a) = J_n(\Gamma_{ns}a) = 0$. Hence we see that

$$\int_0^a rJ_n(\Gamma_{nm}r)J_n(\Gamma_{ns}r)\, dr = 0 \qquad m \neq s \tag{4.52}$$

From the nature of the proof it is clear that (4.52) is also true if Γ_{nm} and Γ_{ns} are chosen such that

$$\frac{dJ_n(\Gamma_{nm}r)}{dr} = \frac{dJ_n(\Gamma_{ns}r)}{dr} = 0 \qquad \text{at } r = a$$

or if Γ_{nm} makes $J_n(\Gamma_{nm}a) = 0$ and Γ_{ns} satisfies $J_n'(\Gamma_{ns}a) = 0$. These orthogonality properties are very similar to those for the sinusoidal functions $\sin nx$ and $\cos mx$ over the range $0 \leq x \leq 2\pi$. When $\Gamma_{nm} = \Gamma_{ns}$, the value of the integral is given by (4.51*b*). The example to be considered now will illustrate the use of the above orthogonality properties.

Example 4.4. Potential in a Cylindrical Region. Consider a cylinder of radius a and length d, as in Fig. 4.6. The end face at $z = d$ is kept at a constant potential V_0, while the remainder of the boundary is kept at zero potential. We wish to determine the potential field Φ within the cylinder.

The solution must be of the form $\Phi = f(r)g(\phi)h(z)$. However, in the present case, Φ is independent of the angle ϕ, and hence $g(\phi)$ is a constant. Since Φ is finite at $r = 0$, the solution for f is simply $J_0(\Gamma r)$. The unmodified Bessel function must be chosen here since we require a function of r that goes to zero at $r = a$, a property that I_0 and K_0 do not possess. The corresponding solution for $h(z)$ is hyperbolic and must be chosen as $\sinh \Gamma z$ in order to satisfy the boundary condition $\Phi = 0$ at $z = 0$.

Since $\Phi = 0$ at $r = a$, the allowed values for the separation constant Γ are the roots Γ_{0m} that make $J_0(\Gamma a) = 0$. The general solution for Φ is thus

$$\Phi = \sum_{m=1}^{\infty} A_m J_0(\Gamma_{0m} r) \sinh \Gamma_{0m} z \tag{4.53}$$

The amplitude constants A_m must be determined so that $\Phi = V_0$ at

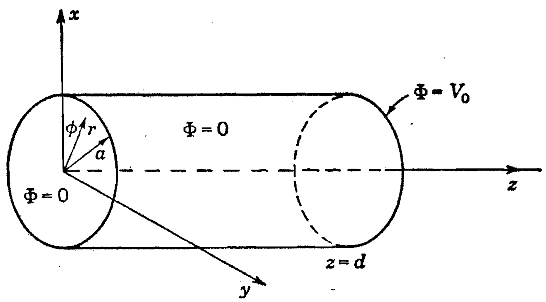

FIG. 4.6. A cylindrical boundary-value problem.

$z = d$, $0 \le r \le a$. Putting $z = d$, we have

$$V_0 = \sum_{m=1}^{\infty} A_m J_0(\Gamma_{0m} r) \sinh \Gamma_{0m} d$$

If we multiply both sides by $rJ_0(\Gamma_{0n} r)\,dr$ and integrate over $0 \le r \le a$, we obtain

$$V_0 \int_0^a rJ_0(\Gamma_{0n} r)\,dr = A_n \sinh \Gamma_{0n} d \int_0^a rJ_0^2(\Gamma_{0n} r)\,dr \tag{4.54}$$

by virtue of the orthogonality property (4.52). From (4.51*b*) we have

$$\int_0^a rJ_0^2(\Gamma_{0n} r)\,dr = \frac{r^2}{2}\left\{\left[\frac{dJ_0(\Gamma_{0n} r)}{d(\Gamma_{0n} r)}\right]^2 + J_0^2(\Gamma_{0n} r)\right\}_0^a$$
$$= \frac{a^2}{2} J_0'^2(\Gamma_{0n} a) = \frac{a^2}{2} J_1^2(\Gamma_{0n} a) \tag{4.55}$$

by utilizing (4.49*a*). Employing the result

$$\int (\Gamma x)^{n+1} J_n(\Gamma x)\,d(\Gamma x) = (\Gamma x)^{n+1} J_{n+1}(\Gamma x) \tag{4.56}$$

gives

$$V_0 \int_0^a rJ_0(\Gamma_{0n} r)\,dr = \frac{V_0}{\Gamma_{0n}^2} (\Gamma_{0n} r) J_1(\Gamma_{0n} r)\Big|_0^a = \frac{V_0}{\Gamma_{0n}} a J_1(\Gamma_{0n} a) \tag{4.57}$$

Combining this result with (4.55) and (4.54) gives the solution for A_n:

$$A_n = \frac{2V_0}{\Gamma_{0n} a J_1(\Gamma_{0n} a) \sinh \Gamma_{0n} d} \tag{4.58}$$

Finally, by substituting into (4.53), we obtain the solution for Φ:

$$\Phi = 2V_0 \sum_{n=1}^{\infty} \frac{J_0(\Gamma_{0n}r) \sinh \Gamma_{0n}z}{\Gamma_{0n}aJ_1(\Gamma_{0n}a) \sinh \Gamma_{0n}d} \tag{4.59}$$

Tables for evaluating this series, i.e., for the roots Γ_{0n} and the values of the Bessel functions, may be found in "Tables of Functions" by Jahnke and Emde.

If we had specified that Φ be equal to zero at $z = 0$ and d, then we should be forced to choose for our functions $h(z)$ the form $\sin(m\pi z/d)$ $(m = 1, 2, \ldots)$. In this case the Bessel function to use must be either $I_0(m\pi r/d)$ or $K_0(m\pi r/d)$. Only I_0 is finite at $r = 0$ and is therefore the only allowed function. Our solution for Φ would now be of the form

$$\Phi = \sum_{m=1}^{\infty} A_m I_0\left(\frac{m\pi r}{d}\right) \sin \frac{m\pi z}{d}$$

At $r = a$ we could specify that Φ be a function of z, say $V(z)$, and by the usual Fourier series analysis determine the coefficients A_m. If Φ at $r = a$ has a ϕ dependence also, then the solution for Φ would be of the form

$$\Phi = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} C_{nm}(\cos n\phi + B_n \sin n\phi) I_n\left(\frac{m\pi r}{d}\right) \sin \frac{m\pi z}{d}$$

In this case a double Fourier series analysis has to be carried out in order to find the coefficients C_{nm} and B_n.

4.3. Spherical Coordinates

Problems such as that of a dielectric sphere placed in a uniform external field are best described in spherical coordinates r, θ, ϕ. With reference to Fig. 4.7, r is the radial coordinate, θ the polar angle, and ϕ the azimuth angle. In spherical coordinates the constant coordinate surfaces are spheres, cones, and planes.

In spherical coordinates Laplace's equation becomes

$$\nabla^2\Phi = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} = 0 \tag{4.60}$$

As before, we assume a product solution of the form $f(r)g(\theta)h(\phi)$. Substituting into (4.60) and dividing by $fgh/(r^2 \sin^2 \theta)$ gives

$$\frac{\sin^2 \theta}{f} \frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{\sin \theta}{g} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial g}{\partial \theta}\right) + \frac{1}{h} \frac{\partial^2 h}{\partial \phi^2} = 0 \tag{4.61}$$

For this equation to be equal to zero for all values of r, θ, and ϕ it is necessary that

$$\frac{\partial^2 h}{\partial \phi^2} + n^2 h = 0 \tag{4.62}$$

where n^2 is a separation constant. The argument is analogous to that used previously; it is only necessary to note that the first two terms in

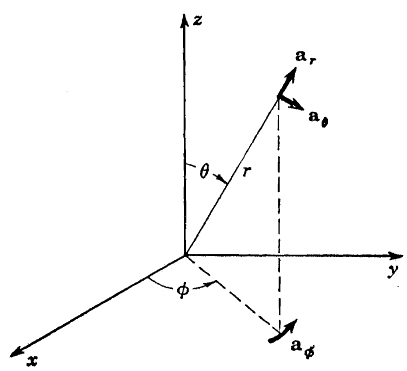

FIG. 4.7. Spherical coordinates.

(4.61) are functions of r and θ only and the last term a function of ϕ alone. For problems involving the whole range $0 \leq \phi \leq 2\pi$, the constant n must be an integer, so that h will be single-valued, i.e., so that $h(2\pi) = h(0)$. The solution to (4.62) is then

$$h(\phi) = C_1 \cos n\phi + C_2 \sin n\phi \tag{4.63}$$

Replacing $\frac{1}{h}\frac{\partial^2 h}{\partial \phi^2}$ by $-n^2$ in (4.61) and dividing through by $\sin^2\theta$ results in

$$\frac{1}{f}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{g \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial g}{\partial \theta}\right) - \frac{n^2}{\sin^2\theta} = 0 \tag{4.64}$$

The first term is a function of r only, while the remaining terms are a function of θ only. For the sum to be equal to zero for all values of r and θ, it is necessary that each term be equal to a constant. Hence we may choose

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) - m(m+1)f = 0 \tag{4.65}$$

where $m(m + 1)$ is the separation constant. The form $m(m + 1)$ is chosen for reasons that will be pointed out later. It is readily verified that the solutions to (4.65) are

$$f(r) = B_1 r^m + B_2 r^{-(m+1)} \tag{4.66}$$

From (4.64) and (4.65) we determine that $g(\theta)$ must satisfy the following differential equation:

$$\frac{d}{d\theta}\left(\sin\theta \frac{dg}{d\theta}\right) + \left[m(m+1)\sin\theta - \frac{n^2}{\sin\theta}\right] g = 0 \tag{4.67}$$

This is Legendre's equation. The standard form of this equation is obtained by making the substitution $\cos\theta = u$:

$$\frac{d}{d\theta} = \frac{d}{du}\frac{du}{d\theta} = -(1-u^2)^{1/2}\frac{d}{du}$$

The equation then becomes

$$\frac{d}{du}(1-u^2)\frac{dg(u)}{du} + \left[m(m+1) - \frac{n^2}{1-u^2}\right] g(u) = 0 \tag{4.68}$$

The solutions to this equation are the associated Legendre functions which we shall study briefly.

For the particular case $n = 0$, the equation becomes

$$\frac{d}{du}(1-u^2)\frac{dg}{du} + m(m+1)g = 0 \tag{4.69}$$

Unless the separation constant is chosen in the form $m(m+1)$ with $m = 0, 1, 2, \ldots$, all the solutions to (4.68) and (4.69) become infinite when either $u = 1$ or $u = -1$, that is, when $\theta = 0, \pi$. These solutions would not be suitable for physical problems that include the polar axis. As a differential equation of second degree, (4.69) has two independent solutions. These are called Legendre functions of the first and second kinds and are designated as $P_m{}^0(u)$ and $Q_m{}^0(u)$, respectively. When m is an integer, $P_m{}^0$ is a finite polynomial in u. However, $Q_m{}^0$ has a singularity at the poles $\theta = 0, \pi$. In the following we shall assume that the polar axis is part of the region of interest and that no singularity is to be expected there, so that Legendre functions of the second kind may be excluded.

When $n \neq 0$, the solutions to (4.68) that remain finite at the poles are associated Legendre polynomials that are designated by the symbol $P_m{}^n(u)$, where m and n are positive integers. The polynomials $P_m{}^n$ are readily obtained from the following generating function:

$$P_m{}^n(u) = \frac{(1-u^2)^{n/2}}{2^m m!}\frac{d^{n+m}(u^2-1)^m}{du^{n+m}} \tag{4.70}$$

Several of the Legendre and associated Legendre polynomials are given below; these may be easily confirmed through the use of (4.70).

$$P_0{}^0 = 1 \tag{4.71a}$$

$$P_1{}^0 = \frac{1}{2}\frac{d}{du}(u^2 - 1) = u = \cos\theta \tag{4.71b}$$

$$P_2{}^0 = \tfrac{3}{2}\cos^2\theta - \tfrac{1}{2} = \tfrac{3}{4}\cos 2\theta + \tfrac{1}{4} \tag{4.71c}$$

$$P_3{}^0 = \tfrac{1}{2}(5\cos^3\theta - 3\cos\theta) \tag{4.71d}$$

$$P_1{}^1 = \sin\theta \tag{4.71e}$$

$$P_2{}^1 = \tfrac{3}{2}\sin 2\theta \tag{4.71f}$$

$$P_m{}^n = 0 \qquad n > m \tag{4.71g}$$

From the differential equation the following orthogonality properties may be proved in a manner similar to that used for the Bessel functions:

$$\int_{-1}^{1} P_m{}^n P_l{}^n \, du = \int_0^\pi P_m{}^n P_l{}^n \sin\theta \, d\theta = 0 \qquad m \neq l \tag{4.72a}$$

$$\int_{-1}^{1} P_m{}^n P_m{}^l \frac{du}{1 - u^2} = \int_0^\pi P_m{}^n P_m{}^l \frac{d\theta}{\sin\theta} = 0 \qquad n \neq l \tag{4.72b}$$

When $m = l$ or $n = l$ we obtain

$$\int_{-1}^{1} (P_m{}^n)^2 \, du = \int_0^\pi [P_m{}^n(\cos\theta)]^2 \sin\theta \, d\theta = \frac{2}{2m+1}\frac{(m+n)!}{(m-n)!} \tag{4.73}$$

By means of these formulas an arbitrary piecewise-continuous function $g(\theta)$ may be expanded into a Fourier-type series in terms of the polynomials $P_m{}^n$.

The general solution to Laplace's equation in spherical coordinates, subject to the assumptions already noted, is now seen to be

$$\Phi(r,\theta,\phi) = \sum_{m=0}^{\infty} \sum_{n=0}^{m} (A_n \cos n\phi + B_n \sin n\phi) \times (C_m r^m + D_m r^{-(m+1)}) P_m{}^n(\cos\theta) \tag{4.74}$$

The sum over n terminates at $n = m$ since $P_m{}^n$ is zero for $n > m$. The coefficients A_n, B_n, C_m, and D_m are determined by the boundary conditions that Φ must satisfy.

Example 4.5. Potential Specified on the Surface of a Sphere. Let the surface of a sphere of radius R be kept at a potential

$$\Phi(R,\theta,\phi) = V_0 \sin\phi \sin\theta = V_0 P_1{}^1 \sin\phi \tag{4.75}$$

the latter since $P_1{}^1 = \sin\theta$. We require a solution for Φ in the interior of the sphere. Since Φ must remain finite at $r = 0$, the coefficent D_m in the general solution (4.74) must be zero. Equating (4.74) for $r = R$

to (4.75) gives

$$\sum_{m=0}^{\infty}\sum_{n=0}^{m}(A_n\cos n\phi+B_n\sin n\phi)C_mR^mP_m{}^n=V_0P_1{}^1\sin\phi \qquad (4.76)$$

If we multiply both sides by $\cos s\phi\,d\phi$ and integrate over $0\le\phi\le 2\pi$, we find that $A_n=0$ since we obtain

$$\sum_{m=0}^{\infty}A_s\pi C_mR^mP_m{}^s=V_0P_1{}^1\int_0^{2\pi}\sin\phi\cos s\phi\,d\phi=0$$

by virtue of the orthogonal properties of the $\cos n\phi$ and $\sin n\phi$ functions. If we multiply (4.76) by $P_l{}^s\sin s\phi\sin\theta\,d\theta\,d\phi$, all terms on the left-hand side except $m=l$, $n=s$ integrate to zero between the limits $0\le\phi\le 2\pi$ and $0\le\theta\le\pi$. At the same time the integral of the right-hand side goes to zero unless $l=1$, $s=1$. Hence all B_n and C_m are zero except B_1 and C_1. For $l=s=1$, we obtain

$$\begin{aligned}\tfrac{4}{3}B_1C_1R\pi&=V_0\int_0^{2\pi}\int_0^{\pi}(P_1{}^1)^2\sin^2\phi\sin\theta\,d\theta\,d\phi\\&=\tfrac{4}{3}V_0\pi\end{aligned}$$

by using (4.73). Hence $B_1C_1=V_0/R$, and the solution for Φ is

$$\Phi=V_0\frac{r}{R}P_1{}^1\sin\phi=V_0\frac{r}{R}\sin\theta\sin\phi \qquad (4.77)$$

Example 4.6. Dielectric Sphere in a Uniform Applied Field. Consider a dielectric sphere of permittivity ϵ, radius R, placed in a uniform external field $E_0\mathbf{a}_z$, as in Fig. 4.8. The applied field may be derived from a potential $\Phi_0=-E_0z=-E_0r\cos\theta=-E_0rP_1{}^0(\cos\theta)$. Since the applied potential is independent of the angle ϕ, the induced potential Φ will also be independent of ϕ. Thus Φ must be of the form

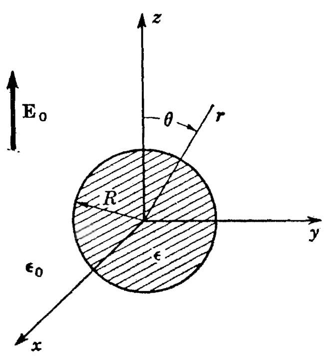

FIG. 4.8. A dielectric sphere in a uniform applied field.

$$\begin{aligned}\Phi&=\sum_{m=0}^{\infty}a_mr^mP_m{}^0(\cos\theta) &\qquad r\le R\\&=\sum_{m=0}^{\infty}b_mr^{-(m+1)}P_m{}^0(\cos\theta) &\qquad r\ge R\end{aligned}$$

This potential is finite at $r=0$ and vanishes at infinity. At $r=R$ the total potential must be continuous; hence

$$\begin{aligned}-E_0RP_1{}^0(\cos\theta)&+\sum_{m=0}^{\infty}a_mR^mP_m{}^0(\cos\theta)\\&=-E_0RP_1{}^0(\cos\theta)+\sum_{m=0}^{\infty}b_mR^{-(m+1)}P_m{}^0(\cos\theta) \qquad (4.78)\end{aligned}$$

Since the functions P_m^0 are orthogonal over the range $0 \leq \theta \leq \pi$, it is seen from (4.78) that

$$a_m = b_m R^{-(2m+1)} \tag{4.79}$$

An additional condition at $r = R$ is that the normal displacement D_r must be continuous. Hence

$$\epsilon \left[-E_0 P_1^0(\cos\theta) + \sum_{m=1}^{\infty} m a_m R^{m-1} P_m^0(\cos\theta) \right] = \epsilon_0 \left[-E_0 P_1^0(\cos\theta) - \sum_{m=0}^{\infty} (m+1) b_m R^{-(m+2)} P_m^0(\cos\theta) \right] \tag{4.80}$$

Again since the functions P_m^0 are a set of orthogonal functions, the coefficients of each term P_m^0 must be equal. Thus we have

$$\epsilon(-E_0 + a_1) = \epsilon_0(-E_0 - 2b_1R^{-3}) \tag{4.81a}$$

$$\epsilon m a_m R^{m-1} = -\epsilon_0(m+1) b_m R^{-(m+2)} \qquad m > 1 \tag{4.81b}$$

Comparing (4.81*b*) with (4.79) shows that both of these equations can hold only if a_m and b_m equal zero for $m > 1$. For $m = 1$ we obtain, from (4.79) and (4.81*a*),

$$a_1 = \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} E_0 \tag{4.82a}$$

$$b_1 = \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} E_0 R^3 \tag{4.82b}$$

In (4.80) the coefficient of P_0^0 on the left-hand side is zero, and hence b_0 must equal zero. From (4.79) it is then seen that $a_0 = 0$ also.

The complete solution for the induced Φ is

$$\Phi = \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} E_0 r P_1^0 = \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} E_0 z \qquad r \leq R \tag{4.83a}$$

$$\Phi = \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} E_0 \cos\theta \frac{R^3}{r^2} \qquad r \geq R \tag{4.83b}$$

Inside the sphere the induced potential gives rise to a uniform field $-(\epsilon - \epsilon_0)E_0/(\epsilon + 2\epsilon_0)$ directed in the z direction. The total potential within the sphere is consequently reduced from the free-space value. Outside the sphere the induced field is a dipole field, which may be considered due to a z-directed dipole of moment p given by

$$p = 4\pi R^3 \epsilon_0 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} E_0 \tag{4.84}$$

and located at the origin. This result follows from the expression

$$\Phi = \frac{p}{4\pi\epsilon_0} \frac{\cos\theta}{r^2}$$

for the potential set up by a dipole of moment p.

Example 4.7. Potential from a Charged Disk. Figure 4.9 illustrates a circular disk of radius a, located in the xy plane and uniformly charged. The potential field Φ set up by the charged disk will have no variation with the azimuth angle ϕ because of the symmetry of the charge distribution about the polar axis. Furthermore, symmetry conditions also require that the potential field be symmetrical about the plane of the disk, that is, symmetrical about $\theta = \pi/2$, and hence $\Phi(\theta) = \Phi(\pi - \theta)$. The potential Φ is finite at the origin, vanishes at infinity, and is independent of ϕ and can therefore be represented by the following expansions:

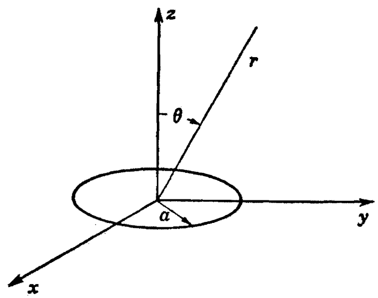

FIG. 4.9. A uniformly charged disk.

$$\Phi = \sum_{m=0}^{\infty} a_m r^m P_m{}^0(\cos\theta) \qquad r \le a,\ \theta < \frac{\pi}{2} \tag{4.85a}$$

$$\Phi = \sum_{m=0}^{\infty} a_m(-1)^m r^m P_m{}^0(\cos\theta) \qquad r \le a,\ \frac{\pi}{2} < \theta < \pi \tag{4.85b}$$

$$\Phi = \sum_{m=0}^{\infty} b_m r^{-(m+1)} P_m{}^0(\cos\theta) \qquad r \ge a,\ \theta < \frac{\pi}{2} \tag{4.85c}$$

$$\Phi = \sum_{m=0}^{\infty} b_m(-1)^m r^{-(m+1)} P_m{}^0(\cos\theta) \qquad r \ge a,\ \frac{\pi}{2} < \theta < \pi \tag{4.85d}$$

The polynomials $P_m{}^0$ for even values of m are even functions of θ about the plane $\theta = \pi/2$, while $P_m{}^0$ for odd values of m are odd functions of θ about the plane $\theta = \pi/2$. In order to satisfy the requirement of even symmetry of Φ about the plane $\theta = \pi/2$, the coefficients a_m and b_m for odd values of m must be replaced by $-a_m$ and $-b_m$ in the region $\pi/2 < \theta < \pi$. For this reason the factor $(-1)^m$ is included in (4.85b) and (4.85d). We shall see later on that all the coefficients b_m for odd values of m vanish. Along the positive z axis the above expansions reduce to the following:

$$\Phi = \sum_{m=0}^{\infty} a_m z^m \qquad z \le a \tag{4.86a}$$

$$\Phi = \sum_{m=0}^{\infty} b_m z^{-(m+1)} \qquad z \ge a \tag{4.86b}$$

since this corresponds to $\cos\theta = 1$ and $P_m{}^0(1) = 1$.

In Example 2.4 the potential along the axis of a uniformly charged disk was found by direct integration of the contribution from each element of charge. This potential was determined to be

$$\Phi = \frac{Q}{2\pi\epsilon_0 a^2}[(a^2 + z^2)^{1/2} - z] \qquad \text{for } z > 0$$

where Q is the total charge on the disk. We may use the binomial theorem to expand Φ into a power series in z and by comparing with (4.86) find the expansion coefficients a_m and b_m. In the region $0 < z < a$ we have

$$\begin{aligned}\Phi &= \frac{Q}{2\pi\epsilon_0 a^2}\left[-z + a\left(1 + \frac{z^2}{a^2}\right)^{1/2}\right] \\ &= \frac{Q}{2\pi\epsilon_0 a^2}\left[-z + a + \frac{a}{2}\left(\frac{z}{a}\right)^2 - \frac{a}{8}\left(\frac{z}{a}\right)^4 + \frac{a}{16}\left(\frac{z}{a}\right)^6 - \cdots\right] \end{aligned} \tag{4.87a}$$

while for $z > a$ a similar expansion gives

$$\begin{aligned}\Phi &= \frac{Q}{2\pi\epsilon_0 a^2}\left[-z + z\left(1 + \frac{a^2}{z^2}\right)^{1/2}\right] \\ &= \frac{Q}{2\pi\epsilon_0 a^2}\left[\frac{z}{2}\left(\frac{a}{z}\right)^2 - \frac{z}{8}\left(\frac{a}{z}\right)^4 + \frac{z}{16}\left(\frac{a}{z}\right)^6 - \cdots\right] \end{aligned} \tag{4.87b}$$

By comparing (4.87) with (4.86) we find, by equating coefficients of like powers of z, that

$$\begin{aligned}
&a_0 = \frac{Q}{2\pi\epsilon_0 a} \qquad a_1 = -\frac{Q}{2\pi\epsilon_0 a^2} \\
&a_2 = \frac{Q}{4\pi\epsilon_0 a^3} \qquad a_3 = a_5 = a_7 = \cdots = 0 \\
&a_4 = -\frac{Q}{16\pi\epsilon_0 a^5} \\
&\cdots\cdots\cdots \\
&b_0 = \frac{Q}{4\pi\epsilon_0} \qquad b_2 = -\frac{Qa^2}{16\pi\epsilon_0} \qquad b_4 = \frac{Qa^4}{32\pi\epsilon_0} \\
&b_1 = b_3 = b_5 = \cdots = 0 \\
&\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

As anticipated from symmetry conditions, all the coefficients b_m for m odd are zero. In the region $r > a$ the potential Φ must be a continuous function of θ with even symmetry about the plane $\theta = \pi/2$. This means that $\partial\Phi/\partial\theta$ is zero at $\theta = \pi/2$ $(r > a)$. Only the polynomials $P_m{}^0(\cos\theta)$ with m even satisfy these requirements. In the region $r < a$ the potential Φ must again be an even function of θ about the symmetry plane $\theta = \pi/2$. At the same time $\partial\Phi/(r\,\partial\theta)$ cannot vanish at $\theta = \pi/2$ since the normal derivative of Φ must equal $-\rho_s/2\epsilon_0$, where ρ_s is the surface density of charge on the disk, as we found in Example 2.4. Hence at least one odd polynomial $P_m{}^0(\cos\theta)$ has to be present in the expansion of Φ for the region

$r < a$. From the results obtained we see that this term is $a_1 r P_1{}^0(\cos\theta)$ for $\theta < \pi/2$ and $-a_1 r P_1{}^0(\cos\theta)$ for $\pi/2 < \theta < \pi$.

For the even polynomials $\partial P_m{}^0/\partial\theta$ equals zero at $\theta = \pi/2$. Consequently, for $r < a$,

$$\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\bigg|_{\theta=\pi/2} = a_1 \frac{\partial\cos\theta}{\partial\theta}\bigg|_{\theta=\pi/2} = -a_1 = \frac{\rho_s}{2\epsilon_0} \tag{4.88}$$

The charge density ρ_s is equal to $Q/\pi a^2$, and thus our previous result $a_1 = -Q/2\pi\epsilon_0 a^2$ is verified.

It should be noted that the two expansions in (4.85) for $r > a$ and $r < a$ must be equal at $r = a$, but since the disk separates the region $r < a$ into two parts and the functions $P_m{}^0$ are not orthogonal over the range $0 \leq \theta \leq \pi/2$, the coefficients a_m cannot be found very readily in terms of the b_m.

For any axially symmetric charge distribution for which a power series expansion in z can be found for the potential along the axis, the above method may be used to determine the potential at all points outside the charge region. Since an expansion like (4.85) is a solution to Laplace's equation and gives the right value of potential along the polar axis, it gives a unique solution.

4.4. Solution of Two-dimensional Problems by Conformal Mapping

The theory of functions of a complex variable provides a powerful method for the solution of Laplace's equation in two dimensions. The viewpoint adopted here is one that considers an analytic function of a complex variable $Z = x + jy$ as generating a suitable orthogonal curvilinear coordinate system which is appropriate for the description of the problem being considered. It is assumed that the student has some familiarity with complex numbers and functions of a complex variable. However, in order to provide continuity, a brief review of the basic concepts that will be required is presented first.

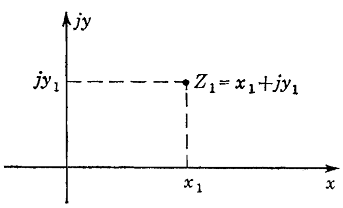

FIG. 4.10. The complex number $x_1 + jy_1$ in the complex Z plane.

The complex variable $Z = x + jy$ is the sum of the real variable x and the product of the imaginary number $j = \sqrt{-1}$ and the real variable y. The complex number $Z_1 = x_1 + jy_1$ is conveniently represented by the point with coordinates x_1, y_1 in the complex Z plane, as in Fig. 4.10. The sum of two complex numbers is the complex number obtained by adding the real parts and the imaginary parts; thus

$$(x_1 + jy_1) + (x_2 + jy_2) = (x_1 + x_2) + j(y_1 + y_2)$$

The product of two complex numbers is formed according to the usual rules of algebra and replacing j^2 by -1, j^3 by $-j$, j^4 by 1, etc. As an example,

$$\begin{aligned}(x_1 + jy_1)(x_2 + jy_2) &= x_1x_2 + jx_1y_2 \\ &\quad + jy_1x_2 + j^2y_1y_2 = (x_1x_2 - y_1y_2) + j(x_1y_2 + x_2y_1)\end{aligned}$$

The complex conjugate of $x_1 + jy_1$ is obtained by replacing j by $-j$. An asterisk is used to denote this operation of taking the complex conjugate; thus

$$(x_1 + jy_1)^* = x_1 - jy_1$$

The quotient of two complex numbers is found by multiplying the numerator and denominator by the conjugate of the denominator; e.g.,

$$\frac{x_1 + jy_1}{x_2 + jy_2} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{(x_2 + jy_2)(x_2 - jy_2)} = \frac{(x_1x_2 + y_1y_2) + j(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

A function of the complex variable Z, say $W = F(Z)$, is called a complex function. An example is $W = Z^2$. The function W will be complex, with a real part u and an imaginary part jv, where u and v are obviously functions of x and y. For the above example,

$$\begin{aligned}W = u + jv = Z^2 &= (x + jy)^2 \\ &= (x^2 - y^2) + 2jxy\end{aligned}$$

and $u = x^2 - y^2$, $v = 2xy$. As the variable Z moves along some curve C in the complex Z plane, the variable $W = F(Z)$ will move along some

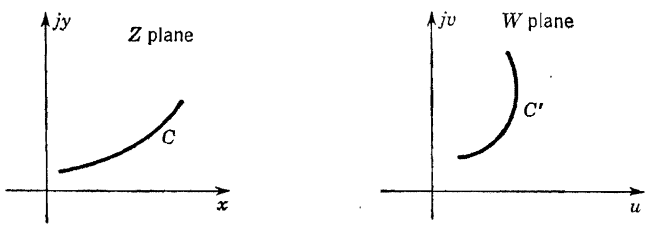

FIG. 4.11. Mapping of a curve C in Z plane into curve C' in W plane.

curve C' in the complex W plane, as illustrated in Fig. 4.11. The curve C' is called the mapping of the curve C.

Of all the possible functions of the complex variable Z, only those functions which have a unique derivative at almost all points in the Z plane are of practical interest. Such functions are known as analytic (or regular) functions. The derivative of $W = F(Z)$ is defined as

$$\frac{dW}{dZ} = \lim_{\Delta Z \to 0} \frac{\Delta W}{\Delta Z} = \lim_{\Delta z \to 0} \frac{W(Z + \Delta Z) - W(\Delta Z)}{\Delta Z} \tag{4.89}$$

If this limit exists and is independent of the direction along which ΔZ approaches zero in the complex Z plane, the function W is said to be analytic at that point. We may readily evaluate dW/dZ under the two conditions where $\Delta Z = \Delta x$ and $\Delta Z = j\,\Delta y$. For the first case we have

$$\frac{dW}{dZ} = \frac{dW}{dx} = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x}$$

while in the second case we have

$$\frac{dW}{dZ} = \frac{dW}{j\,dy} = \frac{\partial u}{j\,\partial y} + \frac{\partial v}{\partial y} = -j\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

If the derivative is to be unique, the two results must be equal, and hence

$$\frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} = -j\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

or

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{4.90}$$

This condition is seen to be a necessary condition, at least, in order for $W = F(Z)$ to be an analytic function. Equations (4.90) are known as the Cauchy-Riemann equations. It is possible to show that if u and v are continuous functions of x and y, the Cauchy-Riemann equations are both necessary and sufficient conditions for $F(Z)$ to be an analytic function. When the derivative exists it may be found by the same rules as are used for functions of a real variable. As examples we have

$$\frac{d\cos Z}{dZ} = -\sin Z$$

$$\frac{dZ^n}{dZ} = nZ^{n-1}$$

$$\frac{de^{2Z}}{dZ} = 2e^{2Z}$$

$$\frac{d\ln Z}{dZ} = \frac{1}{Z}$$

All the above functions are analytic, a result readily verified by showing that the Cauchy-Riemann equations are satisfied. Consider

$$W = \cos Z = \cos(x + jy) = \cos x \cosh y - j\sin x \sinh y$$

for which $u = \cos x \cosh y$, and $v = -\sin x \sinh y$. From (4.90) we obtain

$$\frac{\partial u}{\partial x} = -\sin x \cosh y = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = \cos x \sinh y = -\frac{\partial v}{\partial x}$$

and hence $\cos Z$ is an analytic function.

Coordinate Transformations

In order to appreciate the role played by complex functions in the solution of two-dimensional potential problems, we need to consider some fundamental properties of two-dimensional orthogonal curvilinear coordinates. As an example, if we require a solution for the potential between two infinitely long elliptic cylinders, as illustrated in Fig. 4.12, we would find it very difficult to obtain a suitable solution if we solved Laplace's

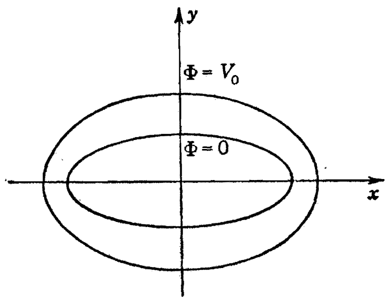

FIG. 4.12. Two concentric elliptic cylinders.

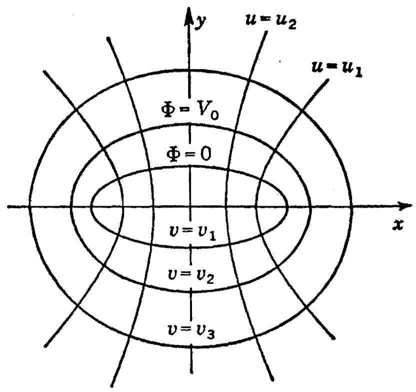

FIG. 4.13. Coordinates uv for elliptic cylinders.

equation in rectangular coordinates. Obviously, we should use elliptic coordinates, so that on each cylinder only one coordinate is variable. In Fig. 4.13 the appropriate coordinates u, v are illustrated. The $v =$ constant curves are ellipses, and the $u =$ constant curves are hyperbolas. Since the boundaries coincide with the $v =$ constant coordinate curves, it follows that on the boundary of a cylinder the potential $\Phi(u,v) = \Phi(u,v_1)$ is a function of u only. For conducting cylinders the appropriate Φ must be independent of u in order that Φ be a constant on the cylinder.

In general, for the problem to be solved we should look for a suitable coordinate system in which the boundaries coincided with constant coordinate curves. Such a set of coordinates u, v are functions of x and y, so that

$$u = u(x,y)$$
$$v = v(x,y)$$

Instead of specifying a point by the coordinates x_1, y_1, we may equally well specify the point by the coordinates $u_1 = u(x_1,y_1)$ and $v_1(x_1,y_1)$. The curves $u =$ constant and $v =$ constant are called coordinate curves.

Along the u = constant curve only v varies, and vice versa. Along the coordinate curves we may construct two unit vectors $\mathbf{a}_u$ and $\mathbf{a}_v$ for the purpose of specifying a vector such as

$$\mathbf{A} = A_u\mathbf{a}_u + A_v\mathbf{a}_v$$

The unit vector $\mathbf{a}_u$ is tangent to the v = constant curves, and $\mathbf{a}_v$ is tangent to the u = constant curves.

When the coordinate curves intersect at right angles, as in Fig. 4.14, the uv coordinates are said to form an orthogonal curvilinear coordinate system. The function ∇u is a vector in the direction of the maximum rate of change of u, and hence

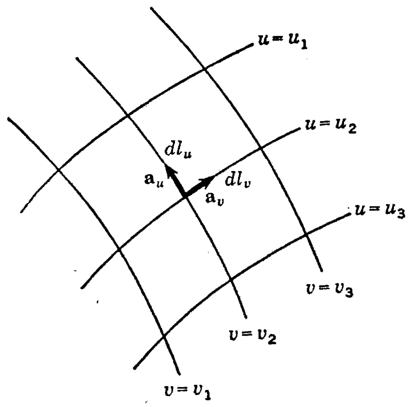

FIG. 4.14. An orthogonal curvilinear coordinate system.

$$\nabla u = |\nabla u|\mathbf{a}_u$$

Similarly, $\quad \nabla v = |\nabla v|\mathbf{a}_v$

If the coordinates form an orthogonal system, then $\mathbf{a}_u$ and $\mathbf{a}_v$ must be orthogonal everywhere. Hence the necessary and sufficient condition for u and v to form an orthogonal coordinate system is that

$$\nabla u \cdot \nabla v = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0 \tag{4.91}$$

The differentials du and dv are not in general measures of length along the coordinate curves. Thus the differentials of length dl_u and dl_v are given by

$$dl_u = h_u\,du \tag{4.92a}$$
$$dl_v = h_v\,dv \tag{4.92b}$$

where h_u and h_v are suitable scale factors. The directional derivative of u along the u coordinate curve is

$$\frac{du}{dl_u} = \nabla u \cdot \mathbf{a}_u = |\nabla u|$$

since ∇u and $\mathbf{a}_u$ are in the same direction. Comparison with (4.92a) shows that

$$dl_u = \frac{du}{|\nabla u|} = h_u\,du$$

and hence
$$h_u = \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]^{-1/2} \tag{4.93a}$$

Similarly,
$$h_v = \left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]^{-1/2} \tag{4.93b}$$

Let us now consider the type of coordinate system generated by a function $W = u + jv = F(Z) = F(x + jy)$. We shall restrict F to be an analytic function, so that the Cauchy-Riemann equations (4.90) hold. The function F gives us two coordinate variables u and v. Using the Cauchy-Riemann equations we can verify that

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} = 0 \tag{4.94}$$

From this and the result of (4.91) we see that u and v form an orthogonal coordinate system. Thus the curves $u =$ constant intersect the curves $v =$ constant at right angles. A further consequence of the Cauchy-Riemann equations is that $h_u = h_v$. This follows since

$$h_v^{-2} = \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 = h_u^{-2}$$

Thus an analytic function $F(Z)$ generates an orthogonal curvilinear coordinate system in which the two scale factors h_u and h_v are equal.

By using the Cauchy-Riemann equations it is easy to show that both u and v satisfy the two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and similarly for v. For this reason u and v are called harmonic functions; i.e., any solution to Laplace's equation is called a harmonic function. Bessel functions and Legendre functions are often referred to as cylindrical and spherical harmonics, respectively, for the same reason.

In the uv coordinates Laplace's equation becomes

$$\frac{\partial}{\partial u}\frac{h_v}{h_u}\frac{\partial \Phi}{\partial u} + \frac{\partial}{\partial v}\frac{h_u}{h_v}\frac{\partial \Phi}{\partial v} = \frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2} = 0 \tag{4.95}$$

since $h_u = h_v$. We obtain the interesting result that $\Phi(x,y)$, which satisfies Laplace's equation

$$\frac{\partial^2 \Phi(x,y)}{\partial x^2} + \frac{\partial^2 \Phi(x,y)}{\partial y^2} = 0$$

also satisfies this equation when Φ is transformed to the uv system, i.e.,

$$\frac{\partial^2 \Phi(u,v)}{\partial u^2} + \frac{\partial^2 \Phi(u,v)}{\partial v^2} = 0$$

Thus in the uv coordinate system it is still only necessary to find a solution to Laplace's equation; furthermore, this will be very much simpler, assuming that an appropriate transformation has been found which makes $u =$ constant or $v =$ constant a boundary surface

The usefulness of the coordinate transformation is often expressed by considering u and v as rectangular coordinates; that is, instead of plotting the u and v coordinate curves on the xy plane, we may distort the uv coordinates into a rectangular grid (W plane) along with the boundaries of the given problem. This means that we transform the curves representing the boundaries in the Z plane into new curves in the W plane. The representation in the W plane is called the conformal mapping of the boundary from the Z plane. The term conformal signifies the property that curves intersecting at right angles in the Z plane map into curves intersecting at right angles in the W plane, a property resulting from the orthogonality of the uv coordinate curves. In the W plane we may now solve for the potential Φ as a function of u and v by treating u and v as rectangular coordinates. The latter solution is possible because surfaces in the Z plane on which $\Phi =$ constant map into the W plane with the same value of constant potential. Similarly, a boundary condition such as $\partial\Phi/\partial n = 0$ is invariant under a conformal transformation. Consequently, the problem in the Z plane is replaced by one in the W plane with identical boundary conditions and the equivalent requirement that $\nabla^2\Phi(u,v) = 0$. But if an appropriate transformation has been found, the problem in the W plane is immeasurably simpler, since we make the boundaries lie along the rectangular $u =$ constant or $v =$ constant lines. Consequently, the solution $\Phi(u,v)$ can be found, and by an inverse transformation $\Phi(x,y)$ is then determined.

The previous remarks will be explained more fully in the course of solution of the following two problems. In the first we consider the case where a constant potential is assigned over the boundary surfaces. In the second the boundary condition $\nabla\Phi \cdot \mathbf{n} = \partial\Phi/\partial n = \rho_s/\epsilon_0$ is given. In this case the boundary value does not remain invariant under a conformal mapping since the scale factor h enters into the expression for $\nabla\Phi$ in the uv coordinate system. The correct technique to be applied is explained in this problem.

Example 4.8. Potential between Two Elliptic Cylinders. We wish to obtain a solution for the potential between two elliptic cylinders with the inner cylinder kept at zero potential and the outer cylinder kept at a potential V_0. The problem is illustrated in Fig. 4.13. In order to solve this problem we must find a complex function $W = F(Z)$ that will generate a uv coordinate system for which the elliptic-cylinder boundaries coincide with constant coordinate curves. There is no direct way in which the required function F may be found. We have to rely on our familiarity with the properties of various analytic functions in order to know which specific function is required. For the present problem the function $W = \cos^{-1} Z$ is a suitable one.

arccos z

For the above function we have $u + jv = \cos^{-1}(x + jy)$ or

$$\cos(u + jv) = \cos u \cosh v - j \sin u \sinh v = x + jy$$

and hence

$$x = \cos u \cosh v \qquad (4.96a)$$
$$y = -\sin u \sinh v \qquad (4.96b)$$

Squaring both sides and adding leads to the result

$$\frac{x^2}{\cosh^2 v} + \frac{y^2}{\sinh^2 v} = \cos^2 u + \sin^2 u = 1 \qquad (4.97)$$

If v is held constant, (4.97) is the equation of an ellipse. The family of

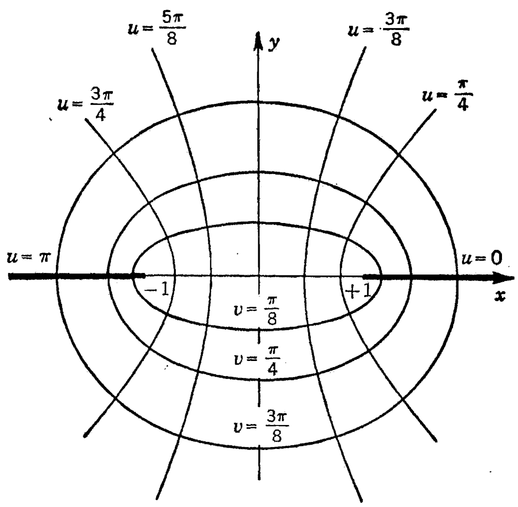

FIG. 4.15. Coordinate curves for inverse cosine function.

curves v = constant are confocal ellipses with foci at ± 1. From (4.96) we may obtain the following equation also by eliminating the variable v:

$$\frac{x^2}{\cos^2 u} - \frac{y^2}{\sin^2 u} = \cosh^2 v - \sinh^2 v = 1 \qquad (4.98)$$

Thus the u = constant curves are a family of confocal hyperbolas which intersect the v = constant curves orthogonally. These coordinates are plotted in Fig. 4.15.

Let the boundaries coincide with the coordinate curves $v = \pi/8$ and $v = \pi/4$. Laplace's equation is

$$\frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2} = 0$$

A simple solution for Φ which is zero for $v = \pi/8$ and equals V_0 for $v = \pi/4$ and is independent of u is

$$\Phi = \frac{8}{\pi}\left(v - \frac{\pi}{8}\right) V_0 \tag{4.99}$$

This is the required solution for Φ since both the boundary conditions and Laplace's equation are satisfied, so that (4.99) is unique. In this

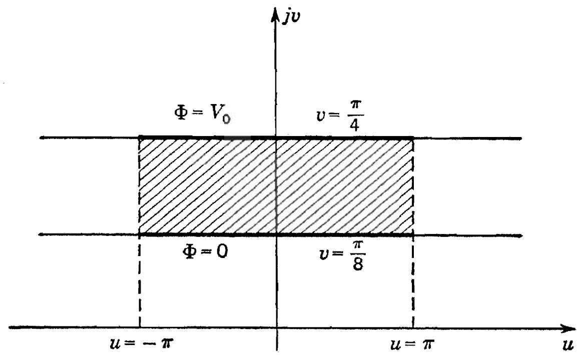

FIG. 4.16. Conformal mapping of Fig. 4.15 into W plane.

particular case Φ is a function of v only. If we solve (4.97) for v, we obtain

$$v = \sinh^{-1}\left[\frac{y^2 + x^2 - 1}{2} \pm \sqrt{\frac{(y^2 + x^2 - 1)^2}{4} + y^2}\right]^{1/2}$$

and this result may be substituted into (4.99) to obtain Φ as a function of x and y.

If we mapped the cylinders $v = \pi/8$ and $v = \pi/4$ into the W plane, our original problem becomes one of finding the potential between two infinite parallel planes separated by a distance $\pi/8$, as in Fig. 4.16. For this problem (4.99) is clearly the solution. As we move around the elliptic cylinder, the coordinate u varies from $-\pi$ to π. The mapping is a periodic one and repeats itself with a periodic 2π in u. The region between the two elliptic cylinders corresponds to the shaded area in Fig. 4.16.

If we are interested in the capacitance C per unit length between the two cylinders, this may be obtained from the equivalent problem in Fig. 4.16. For the latter case C is the capacitance of the finite portion of an infinite-parallel-plane capacitor and is, consequently,

$$C = \frac{2\pi\epsilon_0}{\pi/8} = 16\epsilon_0 \tag{4.100}$$

Note that in view of the infinite geometry there is no fringing field.

The capacitance C is invariant under a conformal mapping and is consequently the desired solution. This result may be proved as follows. The energy stored in the electric field between the two cylinders is given by $W_e = \frac{1}{2}CV_0^2$ per unit length. Now

$$W_e = \frac{\epsilon_0}{2}\int_S |\nabla\Phi|^2\,dx\,dy = \frac{\epsilon_0}{2}\int_{S'}\left[\left(\frac{1}{h_u}\frac{\partial\Phi}{\partial u}\right)^2 + \left(\frac{1}{h_v}\frac{\partial\Phi}{\partial v}\right)^2\right] h_u h_v\,du\,dv$$
$$= \frac{\epsilon_0}{2}\int_{S'}\left[\left(\frac{\partial\Phi}{\partial u}\right)^2 + \left(\frac{\partial\Phi}{\partial v}\right)^2\right] du\,dv \qquad (4.101)$$

since $h_u = h_v$. Hence we may evaluate W_e in the uv plane by treating u and v as rectangular coordinates, and we shall obtain the same result as we get by evaluating the integral in the xy plane. The area S' in the W plane maps into the area S in the Z plane.

Example 4.9. Potential from a Charged Infinitely Long Ribbon. Figure 4.17 illustrates a ribbon extending from -1 to 1 along the x axis. The ribbon may be thought of as a very thin conductor which is charged with a surface charge density

$$\rho = \rho_0 \frac{x}{(1 - x^2)^{1/2}} \qquad (4.102)$$

on both the upper and lower surface. The potential Φ set up by these sources may be determined by means of the inverse cosine function of Example 4.8. When $v = 0$, the elliptic cylinder degenerates into a straight line extending from -1 to 1 along the x axis, and hence $v = 0$ coincides with the surface of the ribbon. When $v = 0$,

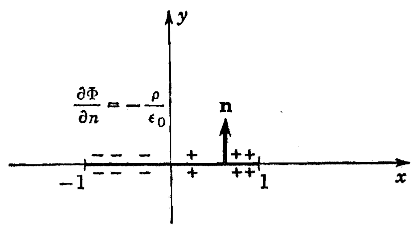

FIG. 4.17. A charged ribbon.

$$x = \cos u \cosh v = \cos u$$

and u therefore varies from 0 to $-\pi$ on the upper surface of the ribbon and from 0 to $+\pi$ on the lower surface. In uv coordinates the charge distribution ρ is given by

$$\rho = \rho_0 \frac{\cos u}{|\sin u|} \qquad (4.103)$$

If $\mathbf{n}$ is the outward normal to the ribbon surface, the boundary condition for Φ on this surface may be specified as

$$E_n = -\nabla\Phi \cdot \mathbf{n} = \frac{\rho}{\epsilon_0}$$

In the uv coordinate system $\mathbf{n} = \mathbf{a}_v$ and

$$\nabla\Phi = \frac{\mathbf{a}_u}{h}\frac{\partial\Phi}{\partial u} + \frac{\mathbf{a}_v}{h}\frac{\partial\Phi}{\partial v}$$

where $h = h_u = h_v$. Hence the boundary condition on Φ becomes

$$\frac{1}{h}\frac{\partial \Phi}{\partial v} = -\frac{\rho}{\epsilon_0} = -\frac{\rho_0}{\epsilon_0}\frac{\cos u}{|\sin u|} \tag{4.104}$$

on the upper surface.

The scale factor h is given by

$$h^{-2} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

since from the Cauchy-Riemann equations $\partial u/\partial y = -\partial v/\partial x$. If the derivative of $W = F(Z)$ with respect to Z is computed with $dZ = dx$, we obtain

$$\frac{dF}{dZ} = \frac{dW}{dZ} = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x}$$

and

$$\left|\frac{dW}{dZ}\right|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

Therefore the scale factor h is given by

$$h = h_u = h_v = \left|\frac{dW}{dZ}\right|^{-1} = \left|\frac{dZ}{dW}\right| \tag{4.105}$$

For the present problem

$$h = \left|\frac{d \cos W}{dW}\right| = |\sin W| = |\sin u \cosh v + j \cosh u \sin v|$$
$$= (\sin^2 u \cosh^2 v + \cosh^2 u \sin^2 v)^{1/2} \tag{4.106}$$

On the ribbon where $v = 0$, we have

$$h = |\sin u|$$

Consequently, the boundary condition (4.104) becomes

$$\frac{\partial \Phi}{\partial v} = -\frac{\rho_0}{\epsilon_0}\cos u \tag{4.107}$$

Our problem may now be stated as follows. Obtain a solution to Laplace's equation

$$\frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2} = 0 \tag{4.108}$$

such that on the surface of the ribbon where $v = 0$ the boundary condition (4.107) is satisfied and also such that as $v \to \infty$, the potential Φ vanishes (note that x and y become infinite only when v does).

Equation (4.107) suggests that Φ will be of the following form: $f(v) \cos u$, where f is a function of v only. If we substitute this into (4.108), we find that f is a solution of

$$\frac{d^2f}{dv^2} - f = 0$$

Hence $f = Ae^{-v} + Be^{v}$, and

$$\Phi = (Ae^{-v} + Be^{v}) \cos u$$

Since Φ must vanish as v becomes infinite, the amplitude coefficient B must be zero. Imposing the boundary condition (4.107) now gives $A = \rho_0/\epsilon_0$; so the complete solution for Φ is

$$\Phi = \frac{\rho_0}{\epsilon_0} e^{-v} \cos u \tag{4.109}$$

The above potential function happens to be the induced potential that is set up if a conducting ribbon is placed in a uniform external field $E_0\mathbf{a}_x$. The applied field may be considered due to an applied potential

$$\Phi_0 = -E_0 x = -E_0 \cos u \cosh v \tag{4.110}$$

The induced potential Φ_i must be such that it cancels the applied electric field along the surface of the ribbon and vanishes at infinity. It works out to be

$$\Phi_i = E_0 e^{-v} \cos u \tag{4.111}$$

At $v = 0$, the total field E_u is

$$E_u = -\frac{1}{h}\frac{\partial(\Phi_0 + \Phi_i)}{\partial u} = \frac{1}{h}(-E_0 \sin u + E_0 \sin u) = 0$$

and vanishes as required. The charge induced on the ribbon is

$$\rho = -\epsilon_0 \frac{1}{h}\frac{\partial \Phi}{\partial v}\bigg|_{v=0} = \frac{\epsilon_0 E_0 \cos u}{|\sin u|}$$

which equals that given by (4.103) if $\epsilon_0 E_0 = \rho_0$.

The induced potential may be written as

$$\begin{aligned}\Phi_i &= E_0(\cosh v - \sinh v) \cos u \\ &= E_0(x - \sinh v \cos u)\end{aligned}$$

by noting that $\cosh v - \sinh v = e^{-v}$. Solving (4.97) for $\sinh v$ and (4.98) for $\cos u$ gives for Φ_i the result

$$\begin{aligned}\Phi_i = E_0 \Bigg\{x - &\left[\frac{x^2 + y^2 - 1}{2} + \sqrt{\frac{(x^2 + y^2 - 1)^2}{4} + y^2}\right]^{1/2} \\ &\times \left[\frac{1 + x^2 + y^2}{2} - \sqrt{\frac{(1 + x^2 + y^2)^2}{4} - x^2}\right]^{1/2}\Bigg\}\end{aligned} \tag{4.112}$$

To obtain this solution by solving Laplace's equation in xy coordinates would be an extremely difficult task. The power of the conformal-mapping or complex-function technique is well illustrated by this example.

4.5. The Schwarz-Christoffel Transformation

The Schwarz-Christoffel transformation is a conformal transformation which will map the real axis in the Z plane into a general polygon in the

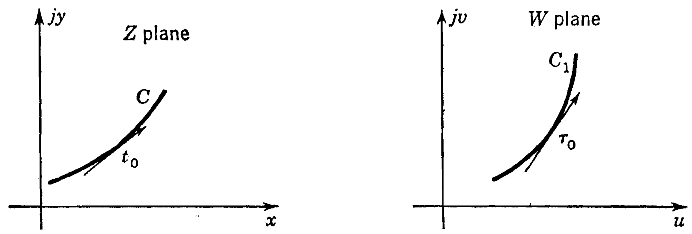

FIG. 4.18. Conformal mapping of C into C_1.

W plane, with the upper half of the Z plane mapping into the region interior to the polygon. To derive the basic transformation we consider first a curve C in the Z plane and its conformal mapping C_1 in the W plane, as illustrated in Fig. 4.18. Let the unit tangent to C in the Z plane be t_0 and the unit tangent to C_1 in the W plane be τ_0, where

$$t_0 = \lim_{\Delta Z \to 0} \frac{\Delta Z}{|\Delta Z|}$$
$$\tau_0 = \lim_{\Delta W \to 0} \frac{\Delta W}{|\Delta W|}$$

If the mapping function is $W = F(Z)$, we have

$$\frac{dW}{dZ} = F'(Z) = \lim_{\Delta Z \to 0} \frac{\Delta W}{\Delta Z} = \lim_{\Delta Z \to 0} \frac{\Delta W/|\Delta W|}{\Delta Z/|\Delta Z|} \left|\frac{\Delta W}{\Delta Z}\right| = \frac{\tau_0}{t_0} |F'(Z)|$$

and hence
$$\tau_0 = \frac{F'(Z)}{|F'(Z)|} t_0 \tag{4.113}$$

The angle that τ_0 makes with the u axis is given by

$$\angle\tau_0 = \angle t_0 + \angle F'(Z) \tag{4.114}$$

Consider next the following function:

$$F'(Z) = A(Z - x_1)^{-k_1}(Z - x_2)^{-k_2} \cdots (Z - x_N)^{-k_N} = A \prod_{i=1}^{N} (Z - x_i)^{-k_i} \tag{4.115}$$

where A is an arbitrary constant, k_i is a real number, and $x_1 < x_2 < \cdots < x_N$. If the x axis is chosen as the curve C in the Z plane, the angle of the unit tangent to the mapping of C, that is, to C_1, in the W plane will be

$$\angle\tau_0 = \angle F' = \angle A - \sum_{i=1}^{N} k_i \angle(Z - x_i) \tag{4.116}$$

For $x < x_i$ we have $\angle(x - x_i) = \pi$, and for $x > x_i$ we have $\angle(x - x_i) = 0$. Thus as each point x_i is passed in the Z plane, the angle of τ_0 changes

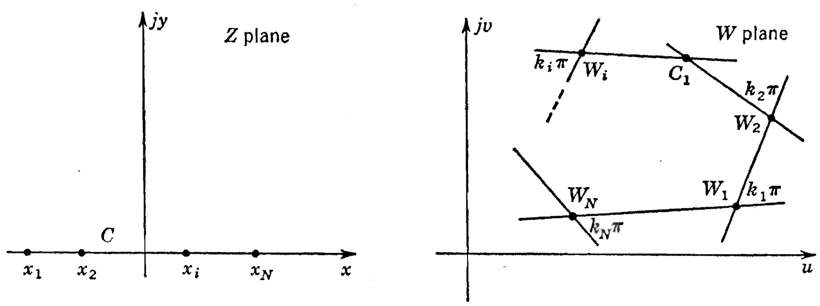

FIG. 4.19. The Schwarz-Christoffel transformation.

discontinuously by an amount $k_i\pi$ and a polygon such as illustrated in Fig. 4.19 is traced out. The exterior angles to the polygon are $k_i\pi$, and these angles must add up to 2π if the polygon is to be closed, and hence

$$\sum_{i=1}^{N} k_i = 2 \tag{4.117}$$

The constant A serves to rotate and magnify the figure in the W plane. The points x_i map into the points $W_i = F(x_i)$. The mapping function is given by

$$W = \int F'(Z)\,dZ + B = A \int \prod_{i=1}^{N} (Z - x_i)^{-k_i}\,dZ + B \tag{4.118}$$

where B is an arbitrary constant which serves to translate the figure in the W plane. The integration of (4.118) is usually not possible unless $|k_i| = 0, \frac{1}{2}, 1, \frac{3}{2}$, or 2. It should be noted that in this transformation there is a factor for each vertex corresponding to a finite value of x but no terms for the points $x = \pm\infty$.

In practice we normally wish to map a given polygon in the W plane into the x axis. This requires the inverse mapping function giving Z as a function of W. Generally, it is difficult to obtain this inverse transformation; so we usually proceed by trial and error to set up a transformation of the form (4.118) which will map the x axis into the given polygon. In this procedure we are aided by the condition that three of the points x_i may be arbitrarily chosen.

Example 4.10. Fringing Capacitance in a Parallel-plate Capacitor. As an example of the use of the Schwarz-Christoffel transformation, the

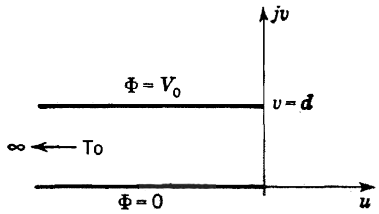

FIG. 4.20. A semi-infinite parallel-plate capacitor.

fringing capacitance C_f at the edge of a semi-infinite parallel-plate capacitor will be determined. The problem is illustrated in Fig. 4.20. In the complex W plane the capacitor boundaries are obtained by letting the points W_1 and W_3 tend to $-\infty$ in Fig. 4.21. The external angles at W_1, W_2, W_3, and W_4 in the limit become $-\pi$, $-\pi$, π, and $-\pi$, respectively.

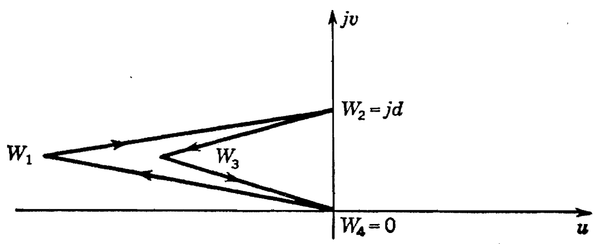

FIG. 4.21. Polygon corresponding to capacitor boundaries.

Let the points W_1, W_2, W_3, W_4 map into the points x_1, x_2, x_3, and x_4 in the Z plane. Our required transformation is thus

$$\begin{aligned} W(Z) &= A_1 \int^Z (Z - x_1)(Z - x_2)(Z - x_3)^{-1}(Z - x_4)\, dZ + B \\ &= A \int^Z \left(1 - \frac{Z}{x_1}\right)(Z - x_2)(Z - x_3)^{-1}(Z - x_4)\, dZ + B \end{aligned}$$

where $A = -A_1 x_1$, and A, B are complex constants.

To cover the complete x axis we now let x_1 tend to minus infinity and hence obtain

$$W(Z) = A \int^Z \frac{(Z - x_2)(Z - x_4)}{Z - x_3}\, dZ + B$$

Since we are free to choose three of the x_i's, let $x_2 = -1$, $x_3 = 0$, and $x_4 = 1$; thus

$$W = A \int^Z \frac{Z^2 - 1}{Z} dZ + B$$

which may be integrated to give

$$W = A\left(\frac{Z^2}{2} - \ln Z\right) + B \tag{4.119}$$

When $W = W_2 = jd$, $Z = -1$, and when $W = W_4 = 0$, $Z = 1$; so

$$jd = A(\tfrac{1}{2} - \ln e^{j\pi}) + B$$
$$0 = A(\tfrac{1}{2}) + B$$

where $e^{j\pi}$ is written for -1. Solving for A and B gives $A = -d/\pi$ and

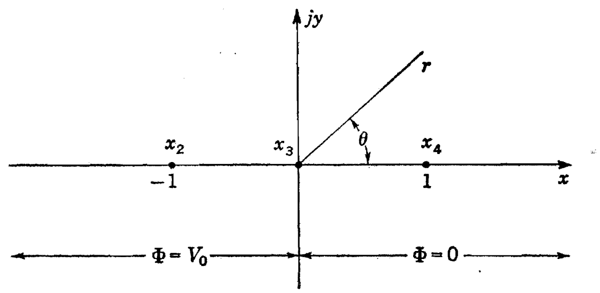

FIG. 4.22. Conformal mapping of capacitor boundaries into Z plane.

$B = d/2\pi$. Our final transformation is

$$W = \frac{d}{\pi}\left(\frac{1 - Z^2}{2} + \ln Z\right) \tag{4.120}$$

and is illustrated in Fig. 4.22.

In the xy plane the solution for Φ is clearly

$$\Phi = V_0 \frac{\theta}{\pi} = \frac{V_0}{\pi} \tan^{-1} \frac{y}{x} \tag{4.121}$$

The charge density on the plane $x < 0$ is given by

$$\rho_s = -\epsilon_0 \frac{\partial \Phi}{\partial n} = \epsilon_0 \frac{1}{r} \frac{\partial \Phi}{\partial \theta} = \frac{\epsilon_0 V_0}{|x|\pi} \tag{4.122}$$

The charge density on the plane $x > 0$ is the negative of (4.122). The total charge on each capacitor plate is, of course, infinite, since each plate is infinite. Let us therefore find the charge $Q(l)$ per unit length for a width l of the lower plate. When $W = -l$ on the inside of the lower plate, let $x = X_1$, where $0 < X_1 < 1$, and when $W = -l$ on the outside of the lower plate, let $x = X_2$, where $X_2 > 1$. For $Q(l)$ we have [$Q(l)$ is taken to be the magnitude of the charge]

$$Q(l) = \frac{\epsilon_0 V_0}{\pi} \int_{X_1}^{X_2} \frac{dx}{x} = \frac{\epsilon_0 V_0}{\pi} (\ln X_2 - \ln X_1)$$

The values of X_1 and X_2 may be found in terms of l from the transformation (4.120). We have

$$\begin{aligned} -l &= \frac{d}{\pi}\left(\frac{1 - X^2}{2} + \ln X\right) = \frac{d}{\pi}\left(\frac{1 - X_1^2}{2} + \ln X_1\right) \\ &= \frac{d}{\pi}\left(\frac{1 - X_2^2}{2} + \ln X_2\right) \end{aligned} \tag{4.123}$$

If we choose $l \gg d$, we can obtain a good approximate solution to these transcendental equations. For X_1 we must have $0 < X_1 < 1$, and in particular for $l \gg d$, X_1 is very small. Hence $(1 - X_1^2)/2$ is negligible compared with $\ln X_1$ and $\ln X_1 \approx -\pi l/d$. For X_2 we must have $X_2 \gg 1$ when $l \gg d$, and we may then neglect the term $\frac{1}{2} + \ln X_2$ to get $X_2^2 \approx 2l\pi/d$ and $\ln X_2 = \frac{1}{2} \ln (2l\pi/d)$. The total charge $Q(l)$ is consequently given by

$$Q(l) = \frac{\epsilon_0 V_0}{\pi}\left(\frac{1}{2} \ln \frac{2l\pi}{d} + \frac{\pi l}{d}\right) \tag{4.124}$$

If there were no fringing field, the charge density on each plate would be constant and equal to $\epsilon_0 V_0/d$. For a width l, the total charge per unit length on the bottom plate would be

$$Q_0(l) = \frac{\epsilon_0 V_0 l}{d} \tag{4.125}$$

The additional charge $Q - Q_0$ gives rise to the fringing capacitance C_f. We therefore find that

$$C_f = \frac{Q - Q_0}{V_0} = \frac{\epsilon_0}{2\pi} \ln \frac{2l\pi}{d} \tag{4.126}$$

for a width l of the capacitor plate, per unit length.

As a practical application of this result consider a parallel-plate capacitor of width and length equal to b and with a spacing $d \ll b$. The parallel-plate capacitance is $\epsilon_0 b^2/d$. A first-order correction to this may be obtained by using (4.126) for the fringing capacitance per edge and choosing $l = b/2$. For the corrected value of capacitance we then obtain

$$C = \frac{\epsilon_0 b^2}{d} + \frac{2b\epsilon_0}{\pi} \ln \frac{\pi b}{d} \tag{4.127}$$

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