

Claim: $\xi^a = \alpha k^a - \beta l^a$, for $\alpha, \beta > 0$, $\Leftrightarrow \eta_k^a = \eta_l^a$ where η_k^a, η_l^a are the unit vectors denoting the spatial directions of k^a, l^a respectively.

\Rightarrow : If $\xi^a = \alpha k^a - \beta l^a$ then $2k_a l^a = \frac{1}{\alpha\beta}$ since $\xi_a \xi^a = -1$ and $k^a k_a = l^a l_a = 0$. Now $k^a + \xi_b k^b \xi^a = k^a - \beta l_b k^b (\alpha k^a - \beta l^a) = \frac{1}{2}(k^a + \frac{\beta}{\alpha} l^a)$ is the spatial part of k^a relative to ξ^a which, upon normalization, yields $\eta_k^a = \frac{k^a}{\beta} + \frac{l^a}{\alpha}$. Similarly $\frac{1}{2}(l^a + \frac{\alpha}{\beta} k^a)$ is the spatial part of l^a relative to ξ^a which, upon normalization, yields $\eta_l^a = \frac{l^a}{\alpha} + \frac{k^a}{\beta} = \eta_k^a$ as desired.

\Leftarrow : The spatial parts, relative to ξ^a , of k^a, l^a are, as above, $k^a + \xi_b k^b \xi^a$ and $l^a + \xi_b l^b \xi^a$ which once normalized become $\eta_k^a = -\frac{k^a}{\xi^b k_b} - \xi^a$ and $\eta_l^a = \frac{l^a}{\xi^b l_b} + \xi^a$; the overall negative sign in η_k^a comes from the fact that $\|k^a + \xi_b k^b \xi^a\| = \sqrt{(\xi^b k_b)^2}$ but $\xi^b k_b < 0$ so we must take $\sqrt{(\xi^b k_b)^2} = -\xi^b k_b$ in order for $\|k^a + \xi_b k^b \xi^a\| > 0$. Now if $\eta_l^a = \eta_k^a$ then $\xi^a = -\frac{1}{2} \frac{k^a}{\xi^b k_b} - \frac{1}{2} \frac{l^a}{\xi^b l_b} = \alpha k^a - \beta l^a$ as desired, where $\alpha \equiv -\frac{1}{2\xi^a k_a}$ and $\beta \equiv \frac{1}{2\xi^a l_a}$.