Representations of Lorentz Group

based on S-33

We defined a unitary operator that implemented a Lorentz transformation on a scalar field:

$$U(\Lambda)^{-1}\varphi(x)U(\Lambda) = \varphi(\Lambda^{-1}x)$$

and then a derivative transformed as:

$\bar{x} = \Lambda^{-1}x$

$$U(\Lambda)^{-1}\partial^\mu\varphi(x)U(\Lambda) = \Lambda^\mu{}_\rho\bar{\partial}^\rho\varphi(\Lambda^{-1}x)$$

it suggests, we could define a vector field that would transform as:

$$U(\Lambda)^{-1}A^\mu(x)U(\Lambda) = \Lambda^\mu{}_\rho A^\rho(\Lambda^{-1}x)$$

and a tensor field $B^{\mu\nu}(x)$ that would transform as:

$$U(\Lambda)^{-1}B^{\mu\nu}(x)U(\Lambda) = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma B^{\rho\sigma}(\Lambda^{-1}x)$$

$$U(\Lambda)^{-1}B^{\mu\nu}(x)U(\Lambda) = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma B^{\rho\sigma}(\Lambda^{-1}x)$$

for symmetric $B^{\mu\nu}(x) = B^{\nu\mu}(x)$ and antisymmetric $B^{\mu\nu}(x) = -B^{\nu\mu}(x)$ tensors, the symmetry is preserved by Lorentz transformations.

In addition, the trace $T(x) \equiv g_{\mu\nu}B^{\mu\nu}(x)$ transforms as a scalar:

$g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = g_{\rho\sigma}$

$$U(\Lambda)^{-1}T(x)U(\Lambda) = T(\Lambda^{-1}x)$$

Thus a general tensor field can be written as:

trace

$$B^{\mu\nu}(x) = A^{\mu\nu}(x) + S^{\mu\nu}(x) + \tfrac{1}{4}g^{\mu\nu}T(x)$$

antisymmetric

symmetric and traceless $g_{\mu\nu}S^{\mu\nu} = 0$

where different parts do not mix with each other under LT!

How do we find the smallest (irreducible) representations of the Lorentz group for a field with n vector indices?

Let's start with a field carrying a generic Lorentz index:

$$U(\Lambda)^{-1}\varphi_A(x)U(\Lambda) = L_A{}^B(\Lambda)\varphi_B(\Lambda^{-1}x)$$

matrices that depend on Λ ,
they must obey the group composition rule

$$L_A{}^B(\Lambda')L_B{}^C(\Lambda) = L_A{}^C(\Lambda'\Lambda)$$

we say these matrices form a representation of the Lorentz group.

For an infinitesimal transformation we had:

$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \delta\omega^\mu{}_\nu$

$$U(1+\delta\omega) = I + \tfrac{i}{2}\delta\omega_{\mu\nu}M^{\mu\nu}$$

where the generators of the Lorentz group satisfied:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\Big(g^{\mu\rho}M^{\nu\sigma} - (\mu\leftrightarrow\nu)\Big) - (\rho\leftrightarrow\sigma)$$

Lie algebra of the Lorentz group.

or in components (angular momentum and boost),

$$J_i \equiv \tfrac{1}{2}\varepsilon_{ijk}M^{jk}$$
$$K_i \equiv M^{i0}$$

we have found:

$$\begin{aligned}
[J_i, J_j] &= i\hbar\varepsilon_{ijk}J_k\ ,\\
[J_i, K_j] &= i\hbar\varepsilon_{ijk}K_k\ ,\\
[K_i, K_j] &= -i\hbar\varepsilon_{ijk}J_k
\end{aligned}$$

$$U(1+\delta\omega) = I + \tfrac{i}{2}\delta\omega_{\mu\nu}M^{\mu\nu}$$

In a similar way, for an infinitesimal transformation we also define:

not necessarily hermitian

$$L_A{}^B(1+\delta\omega) = \delta_A{}^B + \tfrac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu})_A{}^B$$

$$U(\Lambda)^{-1}\varphi_A(x)U(\Lambda) = L_A{}^B(\Lambda)\varphi_B(\Lambda^{-1}x)$$

comparing linear terms in $\delta\omega_{\mu\nu}$

and we find:

$$[\varphi_A(x), M^{\mu\nu}] = \mathcal{L}^{\mu\nu}\varphi_A(x) + (S^{\mu\nu})_A{}^B\varphi_B(x)$$

$$\mathcal{L}^{\mu\nu} \equiv \tfrac{1}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)$$

also it is possible to show that $\mathcal{L}^{\mu\nu}$ and $(S^{\mu\nu})_A{}^B$ obey the same commutation relations as the generators

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\Big(g^{\mu\rho}M^{\nu\sigma} - (\mu\leftrightarrow\nu)\Big) - (\rho\leftrightarrow\sigma)$$

How do we find all possible sets of matrices that satisfy ?

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\Big(g^{\mu\rho}M^{\nu\sigma} - (\mu\leftrightarrow\nu)\Big) - (\rho\leftrightarrow\sigma)$$

$$[J_i, J_j] = i\hbar\varepsilon_{ijk}J_k\,,$$
$$[J_i, K_j] = i\hbar\varepsilon_{ijk}K_k\,,$$
$$[K_i, K_j] = -i\hbar\varepsilon_{ijk}J_k$$

the first one is just the usual set of commutation relations for angular momentum in QM:

for given j (0, 1/2, 1,...) we can find three (2j+1)x(2j+1) hermitian matrices $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$ that satisfy the commutation relations and the eigenvalues of $\mathcal{J}_3$ are -j, -j+1, ..., +j.

not related by a unitary transformation

such matrices constitute all of the inequivalent, irreducible representations of the Lie algebra of SO(3)

cannot be made block diagonal by a unitary transformation

equivalent to the Lie algebra of SU(2)

Crucial observation:

$$[J_i, J_j] = i\hbar\varepsilon_{ijk}J_k\,,\quad [J_i, K_j] = i\hbar\varepsilon_{ijk}K_k\,,\quad [K_i, K_j] = -i\hbar\varepsilon_{ijk}J_k$$

$$N_i \equiv \tfrac{1}{2}(J_i - iK_i)$$
$$N_i^\dagger \equiv \tfrac{1}{2}(J_i + iK_i)$$

$$[N_i, N_j] = i\varepsilon_{ijk}N_k\,,\quad [N_i^\dagger, N_j^\dagger] = i\varepsilon_{ijk}N_k^\dagger\,,\quad [N_i, N_j^\dagger] = 0\,.$$

The Lie algebra of the Lorentz group splits into two different SU(2) Lie algebras that are related by hermitian conjugation!

A representation of the Lie algebra of the Lorentz group can be specified by two integers or half-integers:

$$(2n+1, 2n'+1)$$

there are $(2n+1)(2n'+1)$ different components of a representation

they can be labeled by their angular momentum representations:

since $J_i = N_i + N_i^\dagger$, for given n and n' the allowed values of j are

$$|n-n'|, |n-n'|+1, \ldots, n+n'$$

(the standard way to add angular momenta , each value appears exactly once)

The simplest representations of the Lie algebra of the Lorentz group are:

$$(2n+1, 2n'+1)$$

$(1,1) =$ scalar or singlet

$(2,1) =$ left-handed spinor

$(1,2) =$ right-handed spinor

$(2,2) =$ vector

j = 0 and 1

Left- and Right-handed spinor fields

based on S-34

Let's start with a left-handed spinor field (left-handed Weyl field) $\psi_a(x)$:

left-handed spinor index

under Lorentz transformation we have:

$$U(\Lambda)^{-1}\psi_a(x)U(\Lambda) = L_a{}^b(\Lambda)\psi_b(\Lambda^{-1}x)$$

matrices in the (2,1) representation,

that satisfy the group composition rule:

$$L_a{}^b(\Lambda')L_b{}^c(\Lambda) = L_a{}^c(\Lambda'\Lambda)$$

For an infinitesimal transformation we have:

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \delta\omega^\mu{}_\nu$$

$$L_a{}^b(1+\delta\omega) = \delta_a{}^b + \tfrac{i}{2}\delta\omega_{\mu\nu}(S_{\mathrm{L}}^{\mu\nu})_a{}^b$$

$$(S_{\mathrm{L}}^{\mu\nu})_a{}^b = -(S_{\mathrm{L}}^{\nu\mu})_a{}^b$$

$$[S_{\mathrm{L}}^{\mu\nu}, S_{\mathrm{L}}^{\rho\sigma}] = i\Big(g^{\mu\rho}S_{\mathrm{L}}^{\nu\sigma} - (\mu\leftrightarrow\nu)\Big) - (\rho\leftrightarrow\sigma)$$

Using

$$U(1+\delta\omega) = I + \tfrac{i}{2}\delta\omega_{\mu\nu}M^{\mu\nu}$$

we get

$$U(\Lambda)^{-1}\psi_a(x)U(\Lambda) = L_a{}^b(\Lambda)\psi_b(\Lambda^{-1}x)$$

$$[\psi_a(x), M^{\mu\nu}] = \mathcal{L}^{\mu\nu}\psi_a(x) + (S_{\mathrm{L}}^{\mu\nu})_a{}^b\psi_b(x)$$

$$\mathcal{L}^{\mu\nu} \equiv \tfrac{1}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)$$

present also for a scalar field

to simplify the formulas, we can evaluate everything at space-time origin, $x^\mu = 0$.

and since $M^{ij} = \varepsilon^{ijk}J_k$, we have:

$$\varepsilon^{ijk}[\psi_a(0), J_k] = (S_{\mathrm{L}}^{ij})_a{}^b\psi_b(0)$$

standard convention

$$(S_{\mathrm{L}}^{ij})_a{}^b = \tfrac{1}{2}\varepsilon^{ijk}\sigma_k$$

so that for i=1 and j=2:

$(S_{\mathrm{L}}^{12})_a{}^b = \frac{1}{2}\varepsilon^{12k}\sigma_k = \frac{1}{2}\sigma_3$

$(S_{\mathrm{L}}^{12})_1{}^1 = +\frac{1}{2}$, $(S_{\mathrm{L}}^{12})_2{}^2 = -\frac{1}{2}$

$(S_{\mathrm{L}}^{12})_1{}^2 = (S_{\mathrm{L}}^{12})_2{}^1 = 0$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Once we set the representation matrices for the angular momentum operator, those for boosts $K_k = M^{k0}$ follow from:

$$N_i \equiv \tfrac{1}{2}(J_i - iK_i)$$
$$N_i^\dagger \equiv \tfrac{1}{2}(J_i + iK_i)$$

$$J_k = N_k + N_k^\dagger$$
$$K_k = i(N_k - N_k^\dagger)$$

$N_k^\dagger$ do not contribute when acting on a field in (2,1) representation and so the representation matrices for K_k are i times those for J_k :

$$(S_{\mathrm{L}}^{ij})_a{}^b = \tfrac{1}{2}\varepsilon^{ijk}\sigma_k$$

$$(S_{\mathrm{L}}^{k0})_a{}^b = \tfrac{1}{2}i\sigma_k$$

Let's consider now a hermitian conjugate of a left-handed spinor field $\psi_a(x)$ (a hermitian conjugate of a (2,1) field should be a field in the (1,2) representation) = right-handed spinor field (right-handed Weyl field)

$$[\psi_a(x)]^\dagger = \psi^\dagger_{\dot a}(x)$$

we use dotted indices to distinguish (2,1) from (1,2)!

under Lorentz transformation we have:

$$U(\Lambda)^{-1}\psi^\dagger_{\dot a}(x)U(\Lambda) = R_{\dot a}{}^{\dot b}(\Lambda)\psi^\dagger_{\dot b}(\Lambda^{-1}x)$$

matrices in the (1,2) representation,

that satisfy the group composition rule:

$$R_{\dot a}{}^{\dot b}(\Lambda')R_{\dot b}{}^{\dot c}(\Lambda) = R_{\dot a}{}^{\dot c}(\Lambda'\Lambda)$$

For an infinitesimal transformation we have:

$$R_{\dot a}{}^{\dot b}(1+\delta\omega) = \delta_{\dot a}{}^{\dot b} + \tfrac{i}{2}\delta\omega_{\mu\nu}(S_{\mathrm{R}}^{\mu\nu})_{\dot a}{}^{\dot b}$$

$$(S_{\mathrm{R}}^{\mu\nu})_{\dot a}{}^{\dot b} = -(S_{\mathrm{R}}^{\nu\mu})_{\dot a}{}^{\dot b}$$

$$[\psi_a(x), M^{\mu\nu}] = \mathcal{L}^{\mu\nu}\psi_a(x) + (S_{\mathrm{L}}^{\mu\nu})_a{}^b\psi_b(x)$$

in the same way as for the left-handed field we find:

$$[\psi^\dagger_{\dot a}(0), M^{\mu\nu}] = (S_{\mathrm{R}}^{\mu\nu})_{\dot a}{}^{\dot b}\psi^\dagger_{\dot b}(0)$$

taking the hermitian conjugate,

$$[M^{\mu\nu}, \psi_a(0)] = [(S_{\mathrm{R}}^{\mu\nu})_{\dot a}{}^{\dot b}]^*\psi_b(0)$$

we find:

$$(S_{\mathrm{R}}^{\mu\nu})_{\dot a}{}^{\dot b} = -[(S_{\mathrm{L}}^{\mu\nu})_a{}^b]^*$$

Let's consider now a field that carries two (2,1) indices.
Under Lorentz transformation we have:

$$U(\Lambda)^{-1}C_{ab}(x)U(\Lambda) = L_a{}^c(\Lambda)L_b{}^d(\Lambda)C_{cd}(\Lambda^{-1}x)$$

Can we group 4 components of C into smaller sets that do not mix under Lorentz transformation?

Recall from QM that two spin 1/2 particles can be in a state of total spin 0 or 1:

$$2 \otimes 2 = 1_{\mathrm{A}} \oplus 3_{\mathrm{S}}$$

1 antisymmetric spin 0 state   3 symmetric spin 1 states

Thus for the Lorentz group we have:

$$(2,1) \otimes (2,1) = (1,1)_{\mathrm{A}} \oplus (3,1)_{\mathrm{S}}$$

and we should be able to write:

$$C_{ab}(x) = \varepsilon_{ab}D(x) + G_{ab}(x)$$

$\varepsilon_{ab} = -\varepsilon_{ba}$

$G_{ab}(x) = G_{ba}(x)$

$$C_{ab}(x) = \varepsilon_{ab}D(x) + G_{ab}(x)$$

$\varepsilon_{ab} = -\varepsilon_{ba}$

$\varepsilon_{21} = -\varepsilon_{12} = +1$

D is a scalar

$$U(\Lambda)^{-1}C_{ab}(x)U(\Lambda) = L_a{}^c(\Lambda)L_b{}^d(\Lambda)C_{cd}(\Lambda^{-1}x)$$

$$L_a{}^c(\Lambda)L_b{}^d(\Lambda)\varepsilon_{cd} = \varepsilon_{ab}$$

similar to

$$\Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma g_{\rho\sigma} = g_{\mu\nu}$$

is an invariant symbol of the Lorentz group
(does not change under a Lorentz transformation that acts on all of its indices)

We can use it, and its inverse to raise and lower left-handed spinor indices:

$\varepsilon^{12} = \varepsilon_{21} = +1\,,\qquad \varepsilon^{21} = \varepsilon_{12} = -1\qquad\qquad \varepsilon_{ab}\varepsilon^{bc} = \delta_a{}^c\,,\qquad \varepsilon^{ab}\varepsilon_{bc} = \delta^a{}_c$

to raise and lower left-handed spinor indices:

$$\psi^a(x) \equiv \varepsilon^{ab}\psi_b(x)$$

$\varepsilon_{ab}\varepsilon^{bc} = \delta_a{}^c\,,\qquad \varepsilon^{ab}\varepsilon_{bc} = \delta^a{}_c$

$$\psi^a(x) \equiv \varepsilon^{ab}\psi_b(x)$$

We also have:

$$\psi_a = \varepsilon_{ab}\psi^b = \varepsilon_{ab}\varepsilon^{bc}\psi_c = \delta_a{}^c\psi_c$$

we have to be careful with the minus sign, e.g.:

$$\psi^a = \varepsilon^{ab}\psi_b = -\varepsilon^{ba}\psi_b = -\psi_b\varepsilon^{ba} = \psi_b\varepsilon^{ab}$$

or when contracting indices:

$$\psi^a\chi_a = \varepsilon^{ab}\psi_b\chi_a = -\varepsilon^{ba}\psi_b\chi_a = -\psi_b\chi^b$$

Exactly the same discussion applies to two (1,2) indices:

$$(1,2) \otimes (1,2) = (1,1)_{\mathrm{A}} \oplus (1,3)_{\mathrm{S}}$$

with $\varepsilon_{\dot a\dot b}$ defined in the same way as ε_{ab}: $\varepsilon_{\dot a\dot b} = -\varepsilon_{\dot b\dot a}$, ...

Finally, let's consider a field that carries one undotted and one dotted index; it is in the (2,2) representation (vector):

$$A_{a\dot{a}}(x) = \sigma^{\mu}_{a\dot{a}} A_{\mu}(x)$$

dictionary between the two notations
it is an invariant symbol,
we can deduce its existence from
$(2,1)\otimes(1,2)\otimes(2,2) = (1,1)\oplus\dots$.

more natural way
to write a vector field

A consistent choice with what we have already set for $S^{\mu\nu}_{\mathrm{L}}$ and $S^{\mu\nu}_{\mathrm{R}}$ is:

$$\sigma^{\mu}_{a\dot{a}} = (I, \vec{\sigma})$$

homework

In general, whenever the product of a set of representations includes the singlet, there is a corresponding invariant symbol,

e.g. the existence of $g_{\mu\nu} = g_{\nu\mu}$ follows from

$$(2,2)\otimes(2,2) = (1,1)_{\mathrm{S}} \oplus (1,3)_{\mathrm{A}} \oplus (3,1)_{\mathrm{A}} \oplus (3,3)_{\mathrm{S}}$$

another invariant symbol we will use is completely antisymmetric Levi-Civita symbol:

$$(2,2)\otimes(2,2)\otimes(2,2)\otimes(2,2) = (1,1)_{\mathrm{A}}\oplus\dots$$

$$\varepsilon^{\mu\nu\rho\sigma}$$

$$\varepsilon^{0123} = +1$$

$\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\Lambda^{\rho}{}_{\gamma}\Lambda^{\sigma}{}_{\delta}\varepsilon^{\alpha\beta\gamma\delta}$ is antisymmetric on exchange of any two of its uncontracted indices, and therefore must be proportional to $\varepsilon^{\mu\nu\rho\sigma}$, the constant of proportionality is $\det\Lambda$ which is +1 for proper Lorentz transformations.

Comparing the formula for a general field with two vector indices

$$B^{\mu\nu}(x) = A^{\mu\nu}(x) + S^{\mu\nu}(x) + \tfrac{1}{4}g^{\mu\nu}T(x)$$

trace

antisymmetric

symmetric and traceless $g_{\mu\nu}S^{\mu\nu} = 0$

with

$$(2,2)\otimes(2,2) = (1,1)_{\mathrm{S}} \oplus (1,3)_{\mathrm{A}} \oplus (3,1)_{\mathrm{A}} \oplus (3,3)_{\mathrm{S}}$$

we see that A is not irreducible and, since (3,1) corresponds to a symmetric part of undotted indices,

$$2\otimes 2 = 1_{\mathrm{A}} \oplus 3_{\mathrm{S}}$$

$$C_{ab}(x) = \varepsilon_{ab}D(x) + G_{ab}(x)$$

we should be able to write it in terms of G and its hermitian conjugate.

see Srednicki

Fun with spinor indices

based on S-35

invariant symbol for raising and lowering spinor indices: $\psi^{a}(x) \equiv \varepsilon^{ab}\psi_{b}(x)$

$$\varepsilon^{12} = \varepsilon^{\dot{1}\dot{2}} = \varepsilon_{21} = \varepsilon_{\dot{2}\dot{1}} = +1\,, \qquad \varepsilon^{21} = \varepsilon^{\dot{2}\dot{1}} = \varepsilon_{12} = \varepsilon_{\dot{1}\dot{2}} = -1$$

$$\varepsilon^{ab} = -\varepsilon_{ab} = i\sigma_2$$

another invariant symbol:

$$\sigma^{\mu}_{a\dot{a}} = (I, \vec{\sigma}) \qquad A_{a\dot{a}}(x) = \sigma^{\mu}_{a\dot{a}}A_{\mu}(x)$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Simple identities:

$$\sigma^{\mu}_{a\dot{a}}\sigma_{\mu b\dot{b}} = -2\varepsilon_{ab}\varepsilon_{\dot{a}\dot{b}}$$

$$\varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}\sigma^{\mu}_{a\dot{a}}\sigma^{\nu}_{b\dot{b}} = -2g^{\mu\nu}$$

proportionality constants
from direct calculation

What can we learn about the generator matrices $(S_{\rm L}^{\mu\nu})_a{}^b$ from invariant symbols?

- from $\varepsilon_{ab} = L(\Lambda)_a{}^c L(\Lambda)_b{}^d \varepsilon_{cd}$:

for an infinitesimal transformation we had: $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \delta\omega^\mu{}_\nu$

$$L_a{}^b(1{+}\delta\omega) = \delta_a{}^b + \tfrac{i}{2}\delta\omega_{\mu\nu}(S_{\rm L}^{\mu\nu})_a{}^b$$

and we find:

$$\begin{aligned}\varepsilon_{ab} &= \varepsilon_{ab} + \tfrac{i}{2}\delta\omega_{\mu\nu}\Big[(S_{\rm L}^{\mu\nu})_a{}^c\varepsilon_{cb} + (S_{\rm L}^{\mu\nu})_b{}^d\varepsilon_{ad}\Big] + O(\delta\omega^2)\\ &= \varepsilon_{ab} + \tfrac{i}{2}\delta\omega_{\mu\nu}\Big[-(S_{\rm L}^{\mu\nu})_{ab} + (S_{\rm L}^{\mu\nu})_{ba}\Big] + O(\delta\omega^2)\,.\end{aligned}$$

$$(S_{\rm L}^{\mu\nu})_{ab} = (S_{\rm L}^{\mu\nu})_{ba}$$

similarly:

$$(S_{\rm R}^{\mu\nu})_{\dot a\dot b} = (S_{\rm R}^{\mu\nu})_{\dot b\dot a}$$

- from $\sigma^\rho_{a\dot a} = \Lambda^\rho{}_\tau L(\Lambda)_a{}^b R(\Lambda)_{\dot a}{}^{\dot b}\sigma^\tau_{b\dot b}$:

for infinitesimal transformations we had: $(S_{\rm V}^{\mu\nu})^\rho{}_\tau \equiv \frac{1}{i}(g^{\mu\rho}\delta^\nu{}_\tau - g^{\nu\rho}\delta^\mu{}_\tau)$

$$\begin{aligned}\Lambda^\rho{}_\tau &= \delta^\rho{}_\tau + \tfrac{i}{2}\delta\omega_{\mu\nu}(S_{\rm V}^{\mu\nu})^\rho{}_\tau\,,\\ L_a{}^b(1{+}\delta\omega) &= \delta_a{}^b + \tfrac{i}{2}\delta\omega_{\mu\nu}(S_{\rm L}^{\mu\nu})_a{}^b\,,\\ R_{\dot a}{}^{\dot b}(1{+}\delta\omega) &= \delta_{\dot a}{}^{\dot b} + \tfrac{i}{2}\delta\omega_{\mu\nu}(S_{\rm R}^{\mu\nu})_{\dot a}{}^{\dot b}\,,\end{aligned}$$

isolating linear terms in $\delta\omega_{\mu\nu}$ we have:

$$(g^{\mu\rho}\delta^\nu{}_\tau - g^{\nu\rho}\delta^\mu{}_\tau)\sigma^\tau_{a\dot a} + i(S_{\rm L}^{\mu\nu})_a{}^b\sigma^\rho_{b\dot a} + i(S_{\rm R}^{\mu\nu})_{\dot a}{}^{\dot b}\sigma^\rho_{a\dot b} = 0$$

multiplying by $\sigma_{\rho c\dot c}$ we have:

$$\sigma^\mu_{c\dot c}\sigma^\nu_{a\dot a} - \sigma^\nu_{c\dot c}\sigma^\mu_{a\dot a} + i(S_{\rm L}^{\mu\nu})_a{}^b\sigma^\rho_{b\dot a}\sigma_{\rho c\dot c} + i(S_{\rm R}^{\mu\nu})_{\dot a}{}^{\dot b}\sigma^\rho_{a\dot b}\sigma_{\rho c\dot c} = 0$$

$$\sigma^\mu_{a\dot a}\sigma_{\mu b\dot b} = -2\varepsilon_{ab}\varepsilon_{\dot a\dot b}$$

$$\sigma^\mu_{c\dot c}\sigma^\nu_{a\dot a} - \sigma^\nu_{c\dot c}\sigma^\mu_{a\dot a} + 2i(S_{\rm L}^{\mu\nu})_{ac}\varepsilon_{\dot a\dot c} + 2i(S_{\rm R}^{\mu\nu})_{\dot a\dot c}\varepsilon_{ac} = 0$$

$$\sigma^\mu_{c\dot c}\sigma^\nu_{a\dot a} - \sigma^\nu_{c\dot c}\sigma^\mu_{a\dot a} + 2i(S_{\rm L}^{\mu\nu})_{ac}\varepsilon_{\dot a\dot c} + 2i(S_{\rm R}^{\mu\nu})_{\dot a\dot c}\varepsilon_{ac} = 0$$

multiplying by $\varepsilon^{\dot a\dot c}$ we get: $\varepsilon^{\dot a\dot c}\varepsilon_{\dot a\dot c} = -2$ $\quad\varepsilon^{\dot a\dot c}(S_{\rm R}^{\mu\nu})_{\dot a\dot c} = 0$

$$(S_{\rm L}^{\mu\nu})_{ac} = \tfrac{i}{4}\varepsilon^{\dot a\dot c}(\sigma^\mu_{a\dot a}\sigma^\nu_{c\dot c} - \sigma^\nu_{a\dot a}\sigma^\mu_{c\dot c})$$

similarly, multiplying by ε^{ac} we get:

$$(S_{\rm R}^{\mu\nu})_{\dot a\dot c} = \tfrac{i}{4}\varepsilon^{ac}(\sigma^\mu_{a\dot a}\sigma^\nu_{c\dot c} - \sigma^\nu_{a\dot a}\sigma^\mu_{c\dot c})$$

let's define:

$$\bar\sigma^{\mu\dot a a} \equiv \varepsilon^{ab}\varepsilon^{\dot a\dot b}\sigma^\mu_{b\dot b}$$

$$\sigma^\mu_{a\dot a} = (I,\vec\sigma)$$

$$\bar\sigma^{\mu\dot a a} = (I,-\vec\sigma)$$

we find:

$$(S_{\rm L}^{\mu\nu})_a{}^b = +\tfrac{i}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)_a{}^b$$

$$(S_{\rm R}^{\mu\nu})^{\dot a}{}_{\dot b} = -\tfrac{i}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu)^{\dot a}{}_{\dot b}$$

$_{\dot c}{}^{\dot c}$ $\quad {}^c{}_c$

consistent with our previous choice! (homework)

Convention:

missing pair of contracted indices is understood to be written as:

$$\boxed{{}^c{}_c} \qquad \boxed{{}_{\dot c}{}^{\dot c}}$$

thus, for left-handed Weyl fields we have:

$$\chi\psi = \chi^a\psi_a \quad\text{and}\quad \chi^\dagger\psi^\dagger = \chi^\dagger_{\dot a}\psi^{\dagger\dot a}$$

spin 1/2 particles are fermions that anticommute:

the spin-statistics theorem (later)

$$\chi_a(x)\psi_b(y) = -\psi_b(y)\chi_a(x)$$

and we find:

$$\chi\psi = \chi^a\psi_a = -\psi_a\chi^a = \psi^a\chi_a = \psi\chi$$

$${}_a{}^a = -{}^a{}_a$$

$$\chi\psi = \chi^a\psi_a \quad \text{and} \quad \chi^\dagger\psi^\dagger = \chi^\dagger_{\dot a}\psi^{\dagger\dot a}$$

spin 1/2 particles are fermions that anticommute:

the spin-statistics theorem (later)

$$\chi_a(x)\psi_b(y) = -\psi_b(y)\chi_a(x)$$

and we find:

$$\underline{\chi\psi} = \chi^a\psi_a = -\psi_a\chi^a = \psi^a\chi_a = \underline{\psi\chi}$$

$${}_a{}^a = -{}^a{}_a$$

for hermitian conjugate we find:

$$(\chi\psi)^\dagger = (\chi^a\psi_a)^\dagger = (\psi_a)^\dagger(\chi^a)^\dagger = \psi^\dagger_{\dot a}\chi^{\dagger\dot a} = \psi^\dagger\chi^\dagger$$

as expected if we ignored indices

and similarly:

$$\underline{\psi^\dagger\chi^\dagger} = \underline{\chi^\dagger\psi^\dagger}$$

we will write a right-handed field always with a dagger!

Let's look at something more complicated:

$$\psi^\dagger\bar\sigma^\mu\chi = \psi^\dagger_{\dot a}\bar\sigma^{\mu\dot a c}\chi_c$$

it behaves like a vector field under Lorentz transformations:

$$U(\Lambda)^{-1}[\psi^\dagger\bar\sigma^\mu\chi]U(\Lambda) = \Lambda^\mu{}_\nu[\psi^\dagger\bar\sigma^\nu\chi]$$

evaluated at $\Lambda^{-1}x$

the hermitian conjugate is:

$$\begin{aligned}[\psi^\dagger\bar\sigma^\mu\chi]^\dagger &= [\psi^\dagger_{\dot a}\bar\sigma^{\mu\dot a c}\chi_c]^\dagger \\ &= \chi^\dagger_{\dot c}(\bar\sigma^{\mu a\dot c})^*\psi_a \\ &= \chi^\dagger_{\dot c}\bar\sigma^{\mu\dot c a}\psi_a \\ &= \chi^\dagger\bar\sigma^\mu\psi\,.\end{aligned}$$

$\bar\sigma^\mu = (I, -\vec\sigma)$ is hermitian