Noether Theorem, Noether Charge and All That

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1.0 Transformations

Let G be a Lie group whose action on Minkowski space-time $\left(\mathcal{M}^{(1,3)},\eta\right)$ is formally realized by coordinate transformations

$$x \to \bar{x} = g\,x, \qquad \forall g \in G. \tag{1.1}$$

Infinitesimally, we may write this as

$$\bar{x}^{\mu} = x^{\mu} + \delta x^{\mu}, \tag{1.2}$$

where $\delta x^{\mu} = f^{\mu}(x;\varepsilon)$ are smooth functions of the coordinates and $\varepsilon^{\alpha} \ll 1$, $\alpha = 1,2,...,\dim(\mathrm{G})$, are the infinitesimal group parameters.

We assume that our dynamical variables (the fields) are described by continuous functions $\forall r \in \{1,2,..,n\}, \varphi_r(x) \in \ell^2(\mathcal{M}^{(1,3)},\mathbb{C}^n)$ and transform by finite-dimensional ($n \times n$ matrix) representation ρ of the group G on $\mathcal{M}^{(1,3)}$: ($\rho(G),\mathcal{M}^{(1,3)}$),

$$\varphi_r(x) \to \bar{\varphi}_r(\bar{x}) = D_r{}^s(g)\;\varphi_s(x), \tag{1.3}$$

$$D_r{}^s(g_1\,g_2) = D_r{}^t(g_1)\,D_t{}^s(g_2). \tag{1.4}$$

Close to the identity element of the group, i.e. infinitesimally, we can write

$$D_r{}^s(g) = \delta_r{}^s + \varepsilon^{\alpha}(\Sigma_{\alpha})_r{}^s, \tag{1.5}$$

where the $n \times n$ matrices Σ_{α}'s form a representation of the Lie algebra of the group

$$[\Sigma_{\alpha},\Sigma_{\beta}] = iC_{\alpha\beta}{}^{\gamma}\,\Sigma_{\gamma}. \tag{1.6}$$

Using (1.5), we can rewrite (1.3) as

$$\delta^{*}\varphi_r(x) = \bar{\varphi}_r(\bar{x}) - \varphi_r(x), \tag{1.7}$$

where

$$\delta^{*}\varphi_r(x) \doteq \varepsilon^{\alpha}(\Sigma_{\alpha})_r{}^s\,\varphi_s(x). \tag{1.8}$$

For later use, let us now derive some of the important properties of the variation symbol δ^*. From (1.6) and (1.8), we see that

$$[\delta^*_\alpha, \delta^*_\beta]\varphi_r(x) = iC_{\alpha\beta}{}^\gamma\, \delta_\gamma{}^*\varphi_r(x)\,. \qquad (1.9)$$

Here, we defined the variation symbol δ^*_α by

$$\varepsilon^\alpha\, \delta^*_\alpha(\cdots) = \delta^*(\cdots) \;.$$

Using its definitions (1.7), we can easily show that δ^* acts as derivation on functions

$$\delta^*(f(x)g(x)) = (\delta^* f(x))\, g(x) + f(x)(\delta^* g(x))\,. \qquad (1.10)$$

Since $\delta^*\varphi_r$ is written entirely in terms of the primary fields without time derivatives, it (anti)commutes with $\varphi_s(y)$ at equal time

$$[\delta^*\varphi_r(t,\vec{x}), \varphi_s(t,\vec{y})] = 0\,. \qquad (1.11)$$

To first order in δx^μ we can write

$$\bar{\varphi}_r(\bar{x}) = \bar{\varphi}_r(x) + \delta x^\mu\, \partial_\mu\varphi_r(x)\,. \qquad (1.12)$$

Using this expansion, we can rewrite (1.7) as

$$\delta^*\varphi_r(x) = \delta\varphi_r(x) + \delta x^\mu\, \partial_\mu\varphi_r(x)\,, \qquad (1.13)$$

where

$$\delta\varphi_r(x) \doteq \bar{\varphi}_r(x) - \varphi_r(x)\,, \qquad (1.14)$$

is the infinitesimal change in the functional form of fields. If the field index r is a space-time index taking values in the set $\{\varnothing, \mu, \mu\nu, \mu\nu\rho, ...\}$, in 4-dimensional space-time $\delta\varphi_r(x)$ is nothing but the Lie derivative of a space-time tensor $\varphi_r(x)$ with respect to the infinitesimal transformation (1.2). The transformation considered is called *space-time (symmetry) transformation* if $\delta x^\mu \neq 0$ and *internal (symmetry) transformation* if $\delta x^\mu = 0$; $(\delta^* - \delta)$ is called the *orbital part* of the (symmetry) transformation.

Commuting both sides of (1.13) with $\varphi_s(t,\vec{y})$ (at equal time) and using (1.11), we find

$$[\delta\varphi_r(t,\vec{x}), \varphi_s(t,\vec{y})] = -\delta x^0\, [\dot{\varphi}_r(t,\vec{x}), \varphi_s(t,\vec{y})]\,. \qquad (1.15)$$

And finally, we note that it will be useful to write (1.13) as an operator equation

$$\delta^* = \delta + \delta x^\mu \, \partial_\mu \, , \tag{1.16}$$

with (δ^*, δ) are given by their usual definitions on functions (1.7) and (1.14). Notice that, while δ commutes with derivatives, δ^* does not commute with ∂_σ:

$$[\partial_\sigma, \delta^*] = (\partial_\mu \delta x^\mu)\partial_\sigma \, . \tag{1.17}$$

Exercise (1.1) In D-dimensional Minkowski space-time, show that

$$\delta x^\mu = \in^\mu + \varepsilon x^\mu + \omega^\mu{}_\nu x^\nu + c_\nu \left(2x^\nu x^\mu - \eta^{\mu\nu} x^2\right),$$

is the most general solution to the conformal Killing equation

$$\partial^\mu \left(\delta x^\nu\right) + \partial^\nu \left(\delta x^\mu\right) = \frac{2}{D} \eta^{\mu\nu} \partial_\sigma \left(\delta x^\sigma\right) \; .$$

Solution is given in [1].

Exercise (1.2) Let $f_\mu(x)$ be a solution to the conformal Killing equation

$$\nabla_\mu f_\nu + \nabla_\nu f_\mu = 2\omega(x) g_{\mu\nu} \; .$$

Show that $f_\mu \dot{x}^\mu$ is a constant along the geodesic.

Solution: $\frac{d}{d\tau}\left(f_\mu \dot{x}^\mu\right) = \dot{x}^\sigma \nabla_\sigma \left(f_\mu \dot{x}^\mu\right) = \frac{1}{2}\left(\nabla_\sigma f_\mu + \nabla_\mu f_\sigma\right)\dot{x}^\sigma \dot{x}^\mu = \omega g_{\mu\sigma} \dot{x}^\mu \dot{x}^\sigma = 0$.

2.0 Invariance of the Action and Noether Theorem

Emmy Noether [1882-1935]

The action [$S:(\rho(G),\Omega)\to\mathbb{R}$] is given by the integral

$$S[\varphi_r]=\int_{\Omega}d^4x\,\mathcal{L}(\varphi_r(x),\partial_\mu\varphi_r(x))$$

where $\Omega\subset\mathbb{R}^4$ is an arbitrary, contractible and bounded region in $(\mathcal{M}^{(1,3)},\eta)$. The **Lagrangian density** $\mathcal{L}(x)$ is a local real function of *fundamental* (primary) fields $\varphi_r(x)$, that is, it is constructed from $\varphi_r(x)$ and their derivatives, $\partial_\mu\varphi_r$, *at the same space-time point* x^μ. In order for the canonical formalism to make sense, $\mathcal{L}(x)$ is assumed to contain no second or higher derivatives of $\varphi_r(x)$ and to be at most quadratic in $\partial_\mu\varphi_r$ ($\mathcal{L}:\mathbb{C}^n\times\mathbb{C}^{4n}\to\mathbb{R}$). It is also assumed that the fields are well-behaved and vanish sufficiently rapidly at infinity.

Under the transformation (1.1), a region $D\in\mathcal{M}^{(1,3)}$ is mapped into a new region $\bar{D}$ by point-to-point correspondence. Therefore, infinitesimally $g(\varepsilon)$ induces an infinitesimal change in the region of integration

$$g(\varepsilon):\quad \Omega\to\bar{\Omega}=\Omega+\delta\Omega$$

and maps the action integral into

$$g(\varepsilon): \quad S[\varphi_r] \to S[\overline{\varphi}_r] = \int_{\Omega+\delta\Omega} d^4\overline{x}\,\overline{\mathcal{L}}(\overline{x}) \tag{2.1}$$

where

$$\overline{\mathcal{L}}(\overline{x}) \doteq \mathcal{L}(\overline{\varphi}_r(\overline{x}), \overline{\partial}_\mu \overline{\varphi}_r(\overline{x}))\,.$$

If the action integral is **invariant** (i.e., $S[\overline{\varphi}_r] = S[\varphi_r], \forall \varepsilon^\alpha \text{ and } \varphi_r$) under certain group of transformations (with parameters ε^α), the theory is said to have a **symmetry** corresponding to those transformations. Therefore, in order for the group G to be a symmetry group of our theory, we must have

$$\delta^*\left(\int_\Omega d^4x\,\mathcal{L}(x)\right) = \int_{\Omega+\delta\Omega} d^4\overline{x}\,\overline{\mathcal{L}}(\overline{x}) - \int_\Omega d^4x\,\mathcal{L}(x) = 0\,. \tag{2.2}$$

Changing the dummy integration variable in the second integral, we get

$$\int_{\Omega+\delta\Omega} d^4x\,\overline{\mathcal{L}}(x) - \int_\Omega d^4x\,\mathcal{L}(x) = 0\,. \tag{2.3}$$

Using the relation

$$\overline{\mathcal{L}}(x) = \mathcal{L}(x) + \delta\mathcal{L}(x)\,, \tag{2.4}$$

we can rewrite (2.3) in the form

$$\left(\int_{\Omega+\delta\Omega} d^4x\,\mathcal{L}(x) - \int_\Omega d^4x\,\mathcal{L}(x)\right) + \int_\Omega d^4x\,\delta\mathcal{L}(x) = 0\,. \tag{2.5}$$

We know from elementary calculus that

$$\int_{a+\delta a}^{b+\delta b} dx\, f(x) - \int_a^b dx\, f(x) \approx f(b)\,\delta b - f(a)\,\delta a = \int_a^b dx \frac{d}{dx}\big(f(x)\delta x\big)\,. \tag{2.6}$$

Generalizing this to the 4-dimensional case in (2.5), we find (to first order in δx^μ) that

$$\int_\Omega d^4x\left(\partial_\mu\left(\mathcal{L}\,\delta x^\mu\right) + \delta\mathcal{L}(x)\right) = 0\,. \tag{2.7}$$

The same result can be obtained if we apply the derivation property of δ^* on the left-hand side of (2.2)

$$\delta^*\left(\int_\Omega d^4x\,\mathcal{L}(x)\right) = \int_\Omega \delta^*\left(d^4x\right)\mathcal{L}(x) + \int_\Omega d^4x\,\delta^*\mathcal{L}(x) = 0\,. \tag{2.8}$$

Using the relation

$$\det|1+\in| = \exp\left(\mathrm{Tr}\,\mathrm{In}\,|1+\in|\right) = \exp\left(\mathrm{Tr}\in\right) \approx 1+\mathrm{Tr}\left(\in\right),$$

we can expand the Jacobian of the transformations to first order as

$$\det\left(\frac{\partial\overline{\mathrm{x}}}{\partial\mathrm{x}}\right) = \det\left(\delta_\sigma^\rho + \partial_\sigma\delta x^\rho\right) \approx 1+\mathrm{Tr}\left(\partial_\sigma\delta x^\rho\right) = 1+\partial_\tau\delta x^\tau .$$

Thus

$$d^4\overline{x} = \det\left(\frac{\partial\overline{\mathrm{x}}}{\partial\mathrm{x}}\right)d^4x = d^4x + \partial_\mu(\delta x^\mu)d^4x, \quad \Rightarrow \delta^*(d^4x) = d^4x\,\partial_\mu(\delta x^\mu). \quad (2.9)$$

And applying the operator equation (1.16) to the Lagrangian, gives us

$$\delta^*\mathcal{L} = \delta\mathcal{L} + \delta x^\mu\partial_\mu\mathcal{L}. \quad (2.10)$$

Putting (2.9) and (2.10) in equation (2.8) gives us back equation (2.7).
Since $\Omega\subset\mathbb{R}^4$ is an arbitrary contractible region, the integrand in equation (2.7) must vanishes identically

$$\delta\mathcal{L} + \partial_\mu\left(\mathcal{L}\,\delta x^\mu\right) \equiv 0. \quad (2.11)$$

This is an identity with respect to all its arguments, if the group G is an invariance group of the action integral. It holds for all functions $\varphi_r(x)$, it does not matter whether $\varphi_r(x)$'s are solutions of the Euler-Lagrange equations or not. Since the divergence term is not discarded, the behaviour of $\varphi_r(x)$ on the boundary, $\partial\Omega$, is also irrelevant.
Since $[\delta,\partial_\mu]=0$, we can write

$$\delta\mathcal{L}(x) = \frac{\partial\mathcal{L}}{\partial\varphi_r}\delta\varphi_r + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi_r\right)}\partial_\mu\left(\delta\varphi_r\right). \quad (2.12)$$

Introducing the Euler derivatives

$$\frac{\hat{\delta}\mathcal{L}}{\hat{\delta}\varphi_r} \doteq \frac{\partial\mathcal{L}}{\partial\varphi_r} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial_\mu\varphi_r}\right) = \frac{\delta}{\delta\varphi_r}S[\varphi], \quad (2.13)$$

into equation (2.12) and inserting the resulting equation back into equation (2.11), we arrive at the **Noether Identity**

$$\frac{\hat{\delta}\mathcal{L}}{\hat{\delta}\varphi_r}\delta\varphi_r + \partial_\mu\left(\mathcal{L}\,\delta x^\mu + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi_r\right)}\delta\varphi_r\right) \equiv 0. \quad (2.14)$$

Let us assume the fields $\varphi_r(x)$ are solutions of the Euler-Lagrange equations

$$\frac{\hat{\delta}\mathcal{L}}{\hat{\delta}\varphi_r(x)} = 0, \quad \forall r = 1,2,...,n \,. \tag{2.15}$$

Thus, the Noether identity implies that the object (Noether current)

$$J^\mu(x;\varepsilon) \doteq \mathcal{L}\,\delta x^\mu + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi_r\right)}\delta\varphi_r \,, \tag{2.16}$$

satisfies the conservation law

$$\partial_\mu J^\mu = 0 \,. \tag{2.17}$$

The existence of the conserved (symmetry) current $J^\mu(x)$ is known as the **first Noether theorem**. Of course, physical currents do not depend on the parameters of the symmetry group. Indeed, they can be factored out of equation (2.16) leaving a group index on the physical current

$$J^\mu_{(\alpha)}(x) = \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi_r\right)}\delta_\alpha\varphi_r + \mathcal{L}\,\delta_\alpha x^\mu, \quad \alpha = 1,2,...,\dim(\mathrm{G}) \,. \tag{2.18}$$

Thus, there are as many conserved currents as there are parameters. Notice that the conserved current is not unique. In fact, adding a total divergence of antisymmetric tensor to the Noether current does not spoil the conservation law,

$$\mathcal{J}^\mu{}_{(\alpha)} = J^\mu{}_{(\alpha)} + \partial_\nu\mathcal{F}^{\mu\nu}{}_{(\alpha)}, \quad \mathcal{F}^{\mu\nu}{}_{(\alpha)} = -\mathcal{F}^{\nu\mu}{}_{(\alpha)} \,. \tag{2.19}$$

Quantities like these $\mathcal{F}^{\mu\nu}$ are called super-potentials. They play important role in the construction of the conserved charges in general relativity and other generally covariant theories. Below, we will use this freedom to construct the symmetric energy-momentum tensor out of the non-symmetric canonical tensor.

It will be instructive to give another derivation of the Noether current which is ideologically different from the one we have just done. It consists in comparing the change $\delta\mathcal{L}$ without using the equations of motion (i.e. the *off-shell* variation) with that when the Euler-Lagrange equations are satisfied (i.e. with the *on-shell* variation of $\mathcal{L}$). In this method, the explicit form of $\mathcal{L}$ as well as $\delta\varphi_r$ are assumed known.

According to (2.12), an arbitrary infinitesimal change in the fields induces the following change in the Lagrangian

$$\delta\mathcal{L} = \left(\frac{\partial\mathcal{L}}{\partial\varphi_r} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_r)}\right)\right)\delta\varphi_r + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_r)}\delta\varphi_r\right).$$

Using the equations of motion (i.e. *on-shell*), we find

$$\delta\mathcal{L} = \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu \varphi_r)} \delta\varphi_r \right). \tag{2.20}$$

This is always true, whether or not $\delta\varphi_r$ is a symmetry transformation.
On the other hand, if *without the use of equations of motion* (i.e. just by substituting $\delta\varphi_r$ in (2.12)) the Lagrangian changes by a total divergence of some object Λ^μ,

$$\delta\mathcal{L} = \partial_\mu \Lambda^\mu \ , \tag{2.21}$$

then the action remains unchanged, and the transformation, $\delta\varphi_r$, is a symmetry transformation of the theory.
From (2.19) and (2.20), it follows that the conserved Noether current is

$$J^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu \varphi_r)} \delta\varphi_r - \Lambda^\mu(x) \ . \tag{2.22}$$

Notice that using this method to derive the current, you do not need to know how x^μ transforms under the symmetry group in question. So, how can you distinguish between *internal* and *space-time* symmetries, if you don't have δx^μ? To answer this question we use the fact that adding total divergences to a Lagrangian does not affect the dynamics. Indeed, it is always possible to find a dynamically equivalent Lagrangian $\hat{\mathcal{L}}(x)$ such that

$$\hat{\mathcal{L}}(x) = \mathcal{L}(x) + \partial_\mu \Delta^\mu \ . \tag{2.23}$$

Thus

$$\delta\hat{\mathcal{L}} = \delta\mathcal{L} + \partial_\mu \delta\Delta^\mu = \partial_\mu (\Lambda^\mu + \delta\Delta^\mu) \ . \tag{2.24}$$

So, if $\partial_\mu \Lambda^\mu = 0$, or it is possible to remove it by a suitable choice of Δ^μ, then you are dealing with an *internal* symmetry. Otherwise, it is *space-time* symmetry.

2.1 Translation Invariance and the Stress Energy-Momentum Tensor

Using the relation

$$\delta_\alpha \varphi_r(x) = \delta^*_\alpha \varphi_r(x) - \delta_\alpha x^\nu \, \partial_\nu \varphi_r(x) \, , \tag{2.25}$$

which follows from (1.16) when we factor out the infinitesimal parameters ε^α, we can rewrite (2.18) in the form

$$J^{\mu}{}_{(\alpha)}(x) = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \varphi_r\right)} \delta^*_\alpha \varphi_r - \theta^{\mu}{}_{\nu}(x)\,\delta_\alpha x^\nu\,, \tag{2.26}$$

where

$$\theta^{\mu}{}_{\nu}(x) \doteq \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \varphi_r\right)} \partial_\nu \varphi_r - \delta^{\mu}{}_{\nu}\,\mathcal{L}(x)\,. \tag{2.27}$$

Introducing the canonical conjugate field

$$\pi^r(x) \doteq \frac{\partial \mathcal{L}(x)}{\partial \dot{\varphi}_r(x)}\;, \tag{2.28}$$

into the θ^{00} component, we find

$$\mathcal{H}(x) \doteq \theta^{00}(x) = \pi^r(x)\,\dot{\varphi}_r(x) - \mathcal{L}(x)\,. \tag{2.29}$$

In the classical mechanics of continuous system, $\mathcal{H}(x)$ is nothing but the Hamiltonian density of the system. Therefore, it follows from considerations of covariance that $\theta^{\mu\nu}$ represents the canonical stress energy-momentum tensor of the field. Let us calculate its divergence on shell, i.e. by using the Euler-Lagrange equations of motion,

$$\partial_\mu \theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \varphi_r} \partial^\nu \varphi_r + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)} \partial^\nu \partial_\mu \varphi_r - \partial^\nu \mathcal{L}(\varphi, \partial\varphi, x) = -\eta^{\mu\nu} \frac{\partial \mathcal{L}}{\partial x^\mu}\;.$$

Clearly, this vanishes (on shell) if the Lagrangian does not depend explicitly on x^μ. Thus, the energy-momentum tensor is the conserved Noether current associated with the symmetry group of space-time translations. Indeed, under space-time translations

$$\delta x^\nu = \varepsilon^\nu, \quad \Rightarrow \quad \delta_\alpha x^\nu = \delta^\nu{}_\alpha\;. \tag{2.30}$$

And all fields, regardless of their tensorial character, are invariant,

$$\delta^* \varphi_r(x) = 0\,. \tag{2.31}$$

Putting (2.24) and (2.25) into the defining equation (2.20) of $J^{\mu}{}_{(\alpha)}(x)$, we find

$$J^{\mu}{}_{\alpha}(x) \equiv -\theta^{\mu}{}_{\alpha}(x)\,.$$

Thus, it follows trivially from (2.17) that the energy-momentum tensor is conserved

$$\partial_\mu \theta^{\mu\nu}(x) = 0\,. \tag{2.32}$$

2.2 Lorentz Invariance and the field's Moment Tensor

Under infinitesimal Lorentz transformations

$$\delta x^{\nu} = \eta_{\rho\sigma}\omega^{\nu\sigma}x^{\rho} \ , \tag{2.33}$$

the field transforms according to

$$\delta^{*}\varphi_r(x) = -\frac{i}{2}\omega^{\nu\sigma}\left(\Sigma_{\nu\sigma}\right)_r^{\ s}\varphi_s(x) \ , \tag{2.34}$$

where $\Sigma_{\mu\nu}$'s are the appropriate (spin) transformation matrices for the field φ_r. Using these relations, we can write the conserved current, (2.26), in the form

$$J^{\mu} = -\frac{1}{2}\omega^{\nu\sigma}\left(-i\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\varphi_r)}(\Sigma_{\nu\sigma})_r^{\ s}\varphi_s - (\eta_{\sigma\rho}x^{\rho}\theta^{\mu}{}_{\nu} - \eta_{\nu\rho}x^{\rho}\theta^{\mu}{}_{\sigma})\right).$$

This implies that the moment tensor

$$\begin{aligned}\mathfrak{M}^{\mu}{}_{\nu\sigma} &= (\eta_{\nu\rho}x^{\rho}\theta^{\mu}{}_{\sigma} - \eta_{\sigma\rho}x^{\rho}\theta^{\mu}{}_{\nu}) - i\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\varphi_r)}(\Sigma_{\nu\sigma})_r^{\ s}\varphi_s \\ &= -\mathfrak{M}^{\mu}{}_{\sigma\nu}\end{aligned} \tag{2.35}$$

is conserved. Thus, a field theory is Poincare' invariant *if and only if*

$$\partial_{\mu}\mathfrak{M}^{\mu\nu\sigma} = \theta^{\nu\sigma} - \theta^{\sigma\nu} + \partial_{\mu}\mathcal{S}^{\mu\nu\sigma} = 0 \ . \tag{2.36}$$

The object

$$\mathcal{S}^{\rho\mu\nu}(x) \doteq -i\frac{\partial\mathcal{L}}{\partial(\partial_{\rho}\varphi_r)}(\Sigma^{\mu\nu})_r^{\ s}\varphi_s(x) \ , \tag{2.37}$$

depends entirely on the intrinsic properties (tensorial nature) of the field. In classical field theory, it characterizes the polarization properties of the field. Thus, it corresponds to the spin (angular momentum) of particles described by the quantized field.

For a single component field (scalar), the (spin) matrix Σ vanishes and (2.35) becomes

$$\mathfrak{M}^{\mu\nu\sigma} = x^{\nu}\theta^{\mu\sigma} - x^{\sigma}\theta^{\mu\nu} \ .$$

This relation is very muck like the relation between momentum and orbital angular momentum in the classical mechanics of particle $x_i p_j - x_j p_i = \epsilon_{ijk} L_k$. Therefore, the first term in (2.35),

$$\mathfrak{L}^{\mu\nu\sigma} = x^{\nu}\theta^{\mu\sigma} - x^{\sigma}\theta^{\mu\nu} \ , \tag{2.38}$$

should correspond to some intrinsic "orbital" angular momentum of the field. Thus the moment tensor

$$\mathfrak{M}^{\mu\nu\rho} = \mathfrak{L}^{\mu\nu\rho} + \mathcal{S}^{\mu\nu\rho} \tag{2.39}$$

must be related the total angular momentum tensor of the field.

From (2.35) and (2.36) we can clearly see the connection between the symmetry properties of the canonical energy-momentum tensor $\theta_{\mu\nu}$ and the structure of the total moment tensor $\mathfrak{M}_{\mu\nu\rho}$. For a scalar field, since $\mathcal{S}_{\mu\nu\rho} = 0$, we find that $\theta_{\mu\nu} = \theta_{\nu\mu}$. On the other hand, for arbitrary multi-component field, $\theta_{\mu\nu}$ is not necessarily symmetric. However, the Poincare' invariance condition, (2.36), together with the freedom of adding super-potential to the current, (2.19), allow us to construct a symmetric energy-momentum tensor. Since $\mathcal{S}^{\sigma\mu\nu} = -\mathcal{S}^{\sigma\nu\mu}$, let us seek for a quantity $\mathcal{F}_{\sigma\mu\nu}$ such that

$$\mathcal{S}^{\sigma\mu\nu} = \mathcal{F}^{\sigma\mu\nu} - \mathcal{F}^{\sigma\nu\mu} \quad . \tag{2.40}$$

Putting this in the Poincare' invariance condition (2.36), we find

$$\theta^{\mu\nu} + \partial_\sigma \mathcal{F}^{\sigma\mu\nu} = \theta^{\nu\mu} + \partial_\sigma \mathcal{F}^{\sigma\nu\mu} \quad .$$

Therefore, if we set

$$T^{\mu\nu} \doteq \theta^{\mu\nu} + \partial_\sigma \mathcal{F}^{\sigma\mu\nu} \quad , \tag{2.41}$$

we see that

$$T^{\mu\nu} = T^{\nu\mu} \quad . \tag{2.42}$$

Furthermore the conservation of the canonical energy-momentum tensor, $\partial_\mu \theta^{\mu\nu} = 0$, implies

$$\partial_\mu T^{\mu\nu} = \partial_\mu \partial_\sigma \mathcal{F}^{\sigma\mu\nu} \quad . \tag{2.43}$$

Thus, the conservation of the symmetric tensor $T_{\mu\nu}$ demands (super-potential)

$$\mathcal{F}^{\sigma\mu\nu} = -\mathcal{F}^{\mu\sigma\nu} \tag{2.44}$$

Now, we can use (2.40) and (2.44) to solve for the super-potential

$$\mathcal{F}^{\sigma\mu\nu} = \frac{1}{2}(\mathcal{S}^{\sigma\mu\nu} - \mathcal{S}^{\mu\sigma\nu} + \mathcal{S}^{\nu\mu\sigma}) \quad . \tag{2.45}$$

The conserved tensor $T_{\mu\nu}$ is called the Belinfante symmetric energy-momentum tensor. In terms of this symmetric tensor, the total moment tensor becomes

$$\mathfrak{M}^{\mu\nu\sigma} = (x^\nu T^{\mu\sigma} - x^\sigma T^{\mu\nu}) + (\mathcal{F}^{\sigma\mu\nu} - \mathcal{F}^{\nu\mu\sigma} - \mathcal{S}^{\mu\nu\sigma}) + \partial_\rho \mathcal{P}^{\rho\mu\nu\sigma},$$
$$\mathcal{P}^{\rho\mu\nu\sigma} = \mathcal{F}^{\rho\mu\sigma} x^\nu - \mathcal{F}^{\rho\mu\nu} x^\sigma$$

The quantity in the middle bracket vanishes when (2.40) and (2.44) are used, and if we drop the super-potential $\mathcal{P}$, we arrive at the Belinfante moment tensor

$$\mathcal{M}^{\mu\nu\sigma} = x^\nu T^{\mu\sigma} - x^\sigma T^{\mu\nu} \ . \tag{2.46}$$

Notice that the (internal) spin part has "disappeared" from the total moment tensor whose conservation now follows from the conservation of the **symmetrical** energy-momentum tensor.
The physical importance of the symmetrical (Belinfante) energy momentum tensor follows from the belief that gravitons couple to $T^{\mu\nu}$, and not to $\theta^{\mu\nu}$. Indeed, if the matter theory is minimally coupled to gravity and its action is varied with respect to the metric tensor $g^{\mu\nu}$, the symmetric energy momentum tensor is obtained in the limit $g_{\mu\nu} \to \eta_{\mu\nu}$

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int d^D x \, \mathcal{L}\left(\varphi, \nabla\varphi, g_{\mu\nu}\right) \ . \tag{2.47}$$

With the symmetrical energy momentum tensor, the conserved Poincare' currents take on the compact (Bessel-Hagen [2]) form

$$J_\mu = -T_{\mu\nu} \delta x^\nu \ . \tag{2.48}$$

Since $\delta x^\nu = \in^\nu + \omega^\nu{}_\lambda x^\lambda$, we can rewrite the current in the form

$$-J_\mu = \in^\nu T_{\mu\nu} + \omega^\nu{}_\sigma x^\sigma T_{\mu\nu} = \in^\nu T_{\mu\nu} - \frac{1}{2}\omega^{\nu\sigma}\left(x_\nu T_{\mu\sigma} - x_\sigma T_{\mu\nu}\right) \ , \tag{2.49}$$

or

$$J^\mu = -\left(\in_\nu T^{\mu\nu} - \frac{1}{2}\omega_{\nu\sigma}\mathcal{M}^{\mu\nu\sigma}\right). \tag{2.50}$$

Taking the divergence of the current (2.48), and using the symmetry of the energy momentum tensor, we find

$$-\partial^\mu J_\mu = (\partial^\mu T_{\mu\nu})\delta x^\nu + \frac{1}{2} T_{\mu\nu}(\partial^\mu \delta x^\nu + \partial^\nu \delta x^\mu) = 0 \ .$$

The first term on the right hand side vanishes because of $\partial_\mu T^{\mu\nu} = 0$, and the second term vanishes because Poincare' transformations

$$\delta x^\mu \doteq f^\mu(x) = \in^\mu + \omega^\mu{}_\nu x^\nu \ , \tag{2.51}$$

are the most general solution to the Killing equation (in any number of dimensions)

$$\partial_\mu f_\nu + \partial_\nu f_\mu = 0 \ . \tag{2.52}$$

Now consider the current

$$J_\mu = T_{\mu\nu}\delta x^\nu \ , \tag{2.53}$$

where $T_{\mu\nu}$ is the conserved and symmetric energy momentum tensor, and $\delta x^\nu = f^\nu$ is vector field satisfying the conformal Killing equation in D-dimensional space-time

$$\partial^\mu f^\nu + \partial^\nu f^\mu = \frac{2}{D}\partial_\sigma f^\sigma \eta^{\mu\nu} \ . \tag{2.54}$$

Taking the divergence of the current, we find

$$\partial_\mu J^\mu = \frac{\partial_\sigma f^\sigma}{D} T^\mu_\mu \ . \tag{2.55}$$

Thus, the current (2.53) is conserved provided that the conserved and symmetric energy momentum tensor is traceless $T^\mu_\mu = 0$. Indeed, conformal invariance allows us to further improve the Belinfante energy-momentum tensor so that it is traceless much in the same way that Poincare' invariance allowed us to make the canonical Energy-momentum tensor symmetric [1]. This way, the conserved conformal currents can be written in the compact Bessel-Hagen form (2.53).

2.3 Poincare and Scale Invariance and the Traceless Energy Momentum Tensor

Under scale (dilatation) transformation

$$x^\mu \to \bar{x}^\mu = e^{-\epsilon} x^\mu, \Rightarrow \delta x^\mu = -\epsilon\, x^\mu \ , \tag{2.56}$$

the fields transform according to

$$\varphi_r(x) \to \bar{\varphi}_r(\bar{x}) = e^{\epsilon d}\varphi_r(x), \Rightarrow \delta^* \varphi_r(x) = \epsilon d \varphi_r(x) \ . \tag{2.57}$$

and

$$\delta\varphi_r(x) = \epsilon\left(d + x^\sigma \partial_\sigma\right)\varphi_r(x) \ . \tag{2.58}$$

where the real number d represents the scaling dimension of the field φ_r (we assume all components have the same scaling dimension). In D dimensions, its value for scalar and vector fields, is given by

$$d = \frac{D-2}{2}. \tag{2.59}$$

Using (2.26), we find the *canonical* dilatation current

$$D^{\mu} = \theta^{\mu}{}_{\nu} x^{\nu} + \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\varphi_r)} d\,\varphi_r \ . \tag{2.60}$$

Thus

$$\partial_{\mu} D^{\mu} = \theta^{\mu}_{\mu} + \partial_{\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\varphi_r)} d\,\varphi_r \right). \tag{2.61}$$

Therefore, in scale invariant theories, the trace of the canonical energy-momentum tensor is given by

$$\theta^{\mu}_{\mu} = -\partial_{\sigma}\left(\frac{\partial \mathcal{L}}{\partial(\partial_{\sigma}\varphi_r)} d\,\varphi_r \right). \tag{2.62}$$

We will now show that Poincare' and scale invariance allow us to construct conserved, symmetric and traceless energy momentum tensor. Recall that for any rank-3 tensor $\mathcal{R}^{\sigma\mu\nu} = -\mathcal{R}^{\sigma\nu\mu}$, anti-symmetric in the last two indices, the following combination is a tensor anti-symmetric in the first two indices

$$\mathcal{F}^{\sigma\mu\nu} = \frac{1}{2}(\mathcal{R}^{\sigma\mu\nu} + \mathcal{R}^{\mu\nu\sigma} - \mathcal{R}^{\nu\sigma\mu}) = -\mathcal{F}^{\mu\sigma\nu}. \tag{2.63}$$

Thus, $\partial_{\mu}\partial_{\sigma}\mathcal{F}^{\sigma\mu\nu} = 0$ and the following modified energy-momentum tensor is conserved

$$T^{\mu\nu} = \theta^{\mu\nu} + \frac{1}{2}\partial_{\sigma}(\mathcal{R}^{\sigma\mu\nu} + \mathcal{R}^{\mu\nu\sigma} - \mathcal{R}^{\nu\sigma\mu}). \tag{2.64}$$

Taking the trace, $T^{\sigma}_{\sigma} = \eta_{\mu\nu} T^{\mu\nu}$, we find

$$T^{\sigma}_{\sigma} = \theta^{\sigma}_{\sigma} + \eta_{\mu\nu}\partial_{\sigma}\mathcal{R}^{\mu\nu\sigma} \ . \tag{2.65}$$

Using the condition for scale invariance, (2.62), the trace becomes

$$T^{\mu}_{\mu} = \partial_{\sigma}\left(\eta_{\mu\nu}\mathcal{R}^{\mu\nu\sigma} - \frac{\partial \mathcal{L}}{\partial(\partial_{\sigma}\varphi_r)} d\,\varphi_r \right). \tag{2.66}$$

Recall that, in order to symmetrize the energy-momentum tensor, Poincare' invariance allowed us to choose $\mathcal{R}^{\sigma\mu\nu} = \mathcal{S}^{\sigma\mu\nu}$, see (2.45). So, for Poincare' and scale invariant theories, traceless energy-momentum tensor can be obtained by writing

$$\mathcal{R}^{\mu\nu\sigma} = \mathcal{S}^{\mu\nu\sigma} + \frac{1}{D-1}\left(\eta^{\mu\nu}V^{\sigma} - \eta^{\mu\sigma}V^{\nu}\right), \tag{2.67}$$

where the vector field V^{μ} is determined from (2.66) by setting $T^{\mu}_{\mu} = 0$:

$$\partial_{\sigma}V^{\sigma} = \partial_{\rho}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\rho}\varphi_r)} d\,\varphi_r - \eta_{\mu\nu}\,\mathcal{S}^{\mu\nu\rho}\right). \tag{2.68}$$

Notice that for a scalar field, the spin matrix vanishes. Thus from (2.37) we find $\mathcal{S}^{\mu\nu\sigma} = 0$. Therefore, if we consider the scale invariant field theory

$$\mathcal{L} = \frac{1}{2}\eta_{\mu\nu}\partial^{\mu}\varphi\partial^{\nu}\varphi - \lambda\varphi^{\frac{2D}{D-2}} , \tag{2.69}$$

we find

$$\partial_{\sigma}V^{\sigma} = \frac{D-2}{4}\partial^2\varphi^2, \Rightarrow V^{\sigma} = \frac{D-2}{4}\partial^{\sigma}\varphi^2 . \tag{2.70}$$

Thus, the conserved, symmetric and traceless energy-momentum tensor is

$$\mathcal{T}^{\mu\nu} = T^{\mu\nu} + \frac{D-2}{4(D-1)}\left(\eta^{\mu\nu}\partial^2 - \partial^{\mu}\partial^{\nu}\right)\varphi^2 , \tag{2.71}$$

where $T^{\mu\nu}$ is the symmetric energy-momentum tensor

$$T^{\mu\nu} = \partial^{\mu}\varphi\partial^{\nu}\varphi - \mathcal{L} . \tag{2.72}$$

Exercise (2.1) Show that the energy momentum tensor (2.71), is conserved, $\partial_{\mu}\mathcal{T}^{\mu\nu} = 0$ and traceless, $\eta_{\mu\nu}\mathcal{T}^{\mu\nu} = 0$.

Exercise (2.2) Consider the free Maxwell theory in D-dimensional space-time

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} .$$

Show that the symmetric, conserved energy momentum tensor is given by the Belifante expression in any number of dimensions

$$T^{\mu\nu} = -F^{\mu\sigma}F^{\nu}{}_{\sigma} + \frac{1}{4}\eta^{\mu\nu}F^2 ,$$

but traceless only in 4 dimensions

$$T^{\sigma}_{\sigma} = \left(\frac{D}{4} - 1 \right) F^2 .$$

Exercise (2.3) Show that Maxwell theory is scale invariant in any number of dimensions. Given that

$$\delta A_\sigma = \in \left(x^\mu \partial_\mu + \frac{D-2}{2} \right) A_\sigma .$$

Show that (up to a trivially conserved super potential) the conserved dilation current is given by

$$D^\mu = T^\mu{}_\nu x^\nu + \frac{4-D}{2} F^{\mu\nu} A_\nu \quad .$$

By explicit calculations, show that $\partial_\mu D^\mu = 0$.

2.4 The New Improved CCJ tensor [3] and the Belinfante tensor from the action: fun with super-potentials

Basically, this subsection is about obtaining the results of the last two subsections from varying the action integral. Recall that "on-shell", the variation of the action can be written as

$$\delta S = \int d^4x \, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)} \delta^* \varphi_r - \theta^\mu{}_\nu \, \delta x^\nu \right) \quad . \tag{2.73}$$

To this variation, we will add zero and show that the coefficient of δx^ν is the "new improved" energy-momentum tensor of Callan, Coleman and Jackiw [3].

Consider the functional

$$R = \int d^4x \, \partial_\sigma \Gamma^\sigma(x) \ ,$$

where Γ^μ is a local function of the field and its derivatives to finite order. We vary this using (1.10),

$$\delta^* R = \int \delta^* \left(d^4x \right) \partial \cdot \Gamma + \int d^4x \, \delta^* \left(\partial \cdot \Gamma \right) \ .$$

If in the first integral we make use of (2.9), and in the second integral we use (1.16), we find

$$\delta^* R = \int d^4x \left(\partial\cdot\delta x\right)\left(\partial\cdot\Gamma\right) + \int d^4x\,\delta\left(\partial\cdot\Gamma\right) + \int d^4x \left(\delta x\cdot\partial\right)\left(\partial\cdot\Gamma\right) \ .$$

Combining the first and the third terms and using $[\delta,\partial]=0$ in the second term, we get

$$\delta^* R = \int d^4x\,\partial_\sigma\left(\delta\Gamma^\sigma + \delta x^\sigma\,\partial_\rho\Gamma^\rho\right) \ . \tag{2.74}$$

Using (1.16) again, this can be rewritten as

$$\delta^* R = \int d^4x\,\partial_\sigma\left(\delta^*\Gamma^\sigma + \left(\delta^\sigma_\tau\,\partial_\rho\Gamma^\rho - \delta^\rho_\tau\,\partial_\rho\Gamma^\sigma\right)\delta x^\tau\right) \ . \tag{2.75}$$

Integrating the second term by part, we find

$$\delta^* R = \int d^4x\,\partial_\sigma\left\{\delta^*\Gamma^\sigma - \left(\delta^\sigma_\tau\Gamma^\rho - \delta^\rho_\tau\Gamma^\sigma\right)\partial_\rho\delta x^\tau\right\} + \partial_\sigma\partial_\rho\left(\delta x^\sigma\Gamma^\rho - \delta x^\rho\Gamma^\sigma\right).$$

The second term vanishes identically, and we are left with

.

$$\delta^* R = \int d^4x\,\partial_\sigma\left\{\delta^*\Gamma^\sigma - \left(\delta^\sigma_\tau\Gamma^\rho - \delta^\rho_\tau\Gamma^\sigma\right)\partial_\rho\delta x^\tau\right\} \ . \tag{2.76}$$

Now, we add (2.76) and subtract (2.75) from (2.73)

$$\begin{aligned}\delta^* S = \int d^4x\,\partial_\sigma &\left\{\frac{\partial\mathcal{L}}{\partial(\partial_\sigma\varphi_r)}\delta^*\varphi_r - \left(\delta^\sigma_\tau\Gamma^\rho - \delta^\rho_\tau\Gamma^\sigma\right)\partial_\rho\delta x^\tau\right\} \\ &- \int d^4x\,\partial_\sigma\left\{\left(\theta^\sigma{}_\tau + \delta^\sigma_\tau\partial_\rho\Gamma^\rho - \delta^\rho_\tau\partial_\rho\Gamma^\sigma\right)\delta x^\tau\right\}\end{aligned} \tag{2.77}$$

Let us assume that our symmetry transformations change the action according to

$$\delta^* S = \int d^4x\,\partial_\sigma\Lambda^\sigma(x) \ . \tag{2.78}$$

Now, if we can find Γ^μ such that

$$\frac{\partial\mathcal{L}}{\partial(\partial_\sigma\varphi_r)}\delta^*\varphi_r - \left(\delta^\sigma_\tau\Gamma^\rho - \delta^\rho_\tau\Gamma^\sigma\right)\partial_\rho\delta x^\tau = \Lambda^\sigma(\varphi,\partial\varphi) \ , \tag{2.79}$$

holds, then the Noether current will have the Bessel-Hagen form

$$J^\sigma = T^\sigma{}_\tau\,\delta x^\tau \ , \tag{2.80}$$

with “a new improved energy-momentum tensor” given by

$$T^{\sigma\tau} = \theta^{\sigma\tau} + \partial_\rho\left(\eta^{\sigma\tau}\Gamma^\rho - \eta^{\rho\tau}\Gamma^\sigma\right) \ . \tag{2.81}$$

Exercise (2.4) Consider the free massless scalar field action in four dimensions

$$S = \frac{1}{2}\int d^4x\, \partial_\mu \varphi \partial^\mu \varphi \ .$$

(i) Show that $\delta^* S = 0$, i.e. $\Lambda^\sigma(x) = 0$, under the Lorentz transformations

$$\delta x^\mu = \omega^\mu{}_\nu x^\nu, \quad \delta^* \varphi = 0 \ .$$

Then show that Lorentz invariance implies the following conditions on Γ^μ

$$\partial^\mu \Gamma^\nu = \partial^\nu \Gamma^\mu \ .$$

(ii) Show that $\delta^* S = 0, \Lambda^\mu = 0$, under the scale transformations

$$\delta x^\mu = -\in x^\mu, \quad \delta^* \varphi = \in \varphi \ .$$

Use (2.79) to show that scale invariance implies

$$\Gamma^\sigma = \frac{1}{6}\partial^\sigma \varphi^2 \ .$$

Use this to show that (2.81) leads to the symmetric and traceless energy-momentum tensor.

$$T^{\mu\nu} = \theta^{\mu\nu} + \frac{1}{6}\left(\eta^{\mu\nu}\partial^2 - \partial^\mu \partial^\nu\right)\varphi^2 \ .$$

(iii) Show that the conformal transformations

$$\delta x^\mu = 2x^\mu (c_\sigma x^\sigma) - c^\mu x^2, \quad \delta^* \varphi = 2(c_\sigma x^\sigma)\varphi \ ,$$

change the scalar field action by total divergence

$$\delta^* S = \int d^4x\, \partial_\sigma \Lambda^\sigma = \int d^4x\, \partial_\sigma \left(c^\sigma \varphi^2\right) \ .$$

Use Lorentz invariance and scale invariance restrictions on Γ^μ to show that (2.79) is satisfied for the conformal transformations.

3.0 Canonical Quantization and Equal-Time Commutators Algebra

Before introducing the Noether charge and discuss its properties, we need to set up our quantization scheme. We will work with the canonical quantization rule: **Poisson bracket of functions goes to equal-time (anti)commutator of the corresponding operators**[1]

$$i\{f(t,\mathbf{x}),g(t,\mathbf{y})\}_{PB}\to[\hat{f}(t,\mathbf{x}),\hat{g}(t,\mathbf{y})] \ .$$

However, we will not use the hat to indicate operators. Also, we will not write the time argument explicitly. Unless stated otherwise, all (anti)commutators are considered to be at equal times. So, for any local functional F , we use the following rules

$$i\{F(x),\varphi_r(y)\}_{PB}=-i\frac{\delta F(x)}{\delta\pi^r(y)}=[F(x),\varphi_r(y)] \ , \tag{3.1}$$

$$i\{F(x),\pi^r(y)\}_{PB}=i\frac{\delta F(x)}{\delta\varphi_r(y)}=[F(x),\pi^r(y)] . \tag{3.2}$$

The fundamental equal-time commutation relations follow by putting $F=\varphi_r,\pi^r$:

$$\begin{aligned}&[\varphi_r(x),\pi^s(y)]=i\delta_r^s\,\delta^3(\mathbf{x}-\mathbf{y}),\\&[\varphi_r(x),\varphi_s(y)]=[\pi^r(x),\pi^s(y)]=0.\end{aligned} \tag{3.3}$$

Notice that

$$\pi^r(x)=\frac{\partial}{\partial\dot{\varphi}_r}\mathcal{L}(\varphi,\nabla\varphi,\dot{\varphi}) \ . \tag{3.4}$$

In most cases, the Lagrangian is quadratic in $\dot{\varphi}_r$. Thus, π^r is linear in $\dot{\varphi}_r$. We will assume that the simultaneous linear equations (3.4) can be solved for $\dot{\varphi}_r$. That is we assume that it is possible to represent the “velocity” by some function

$$\dot{\varphi}_r(x)=f_r(\varphi,\nabla\varphi,\pi) \ . \tag{3.5}$$

The “functional” derivative of any local function of the field variables, such as the Lagrangian, will be calculated using the chain rule:

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[1] In ref. [4] I have out-lined the proof of $\lim_{\hbar\to 0}\frac{-i}{\hbar}(\hat{A}\hat{B}-\hat{B}\hat{A})=\frac{\partial A}{\partial x}\frac{\partial B}{\partial p}-\frac{\partial A}{\partial p}\frac{\partial B}{\partial x}$.

$$\frac{\delta\mathcal{L}(x)}{\delta\varphi_r(y)}=\frac{\partial\mathcal{L}(x)}{\partial\varphi_s(x)}\frac{\delta\varphi_s(x)}{\delta\varphi_r(y)}+\frac{\partial\mathcal{L}(x)}{\partial(\partial_j\varphi_s)}\frac{\delta(\partial_j^{(x)}\varphi_s)}{\delta\varphi_r(y)}+\frac{\partial\mathcal{L}(x)}{\partial\dot{\varphi}_s(x)}\frac{\delta\dot{\varphi}_s(x)}{\delta\varphi_r(y)},$$

$$=\frac{\partial\mathcal{L}(x)}{\partial\varphi_r(x)}\delta^3(\mathbf{x}-\mathbf{y})+\frac{\partial\mathcal{L}(x)}{\partial(\partial_j\varphi_r)}\partial_j^{(x)}\delta^3(\mathbf{x}-\mathbf{y})+\pi^s(x)\frac{\delta\dot{\varphi}_s(x)}{\delta\varphi_r(y)},$$

$$\frac{\delta\mathcal{L}(x)}{\delta\pi^r(y)}=\frac{\partial\mathcal{L}(x)}{\partial\dot{\varphi}_s(x)}\frac{\delta\dot{\varphi}_s(x)}{\delta\pi^r(y)}=\pi^s(x)\frac{\delta\dot{\varphi}_s(x)}{\delta\pi^r(y)}\quad .$$

Thus, using (3.1) and (3.2) in the above two equations, we infer the following equal-times commutation relations

$$[\mathcal{L}(x),\varphi_r(y)]=\pi^s(x)[\dot{\varphi}_s(x),\varphi_r(y)]\quad , \tag{3.6}$$

$$[\mathcal{L}(x),\pi^r(y)]=i\delta^3(\mathbf{x}-\mathbf{y})\frac{\partial\mathcal{L}}{\partial\varphi_r}+i\partial_j^{(x)}\delta^3(\mathbf{x}-\mathbf{y})\frac{\partial\mathcal{L}}{\partial(\partial_j\varphi_r)}+\pi^s(x)[\dot{\varphi}_s(x),\pi^r(y)] \tag{3.7}$$

The last two terms in (3.7) can be combined to give

$$[\mathcal{L}(x),\pi^r(y)]=i\frac{\partial\mathcal{L}(x)}{\partial\varphi_r(x)}\delta^3(\mathbf{x}-\mathbf{y})+\frac{\partial\mathcal{L}(x)}{\partial(\partial_\sigma\varphi_s)}[\partial_\sigma\varphi_s(x),\pi^r(y)]\quad . \tag{3.8}$$

We will now use these commutation relations together with the fundamental equal-time commutation relations (3.3) to derive few important equations. Let's start by deriving the so-called Heisenberg equations

$$\begin{aligned}&[i\mathrm{H},\varphi_r(x)]=\dot{\varphi}_r(x),\\&[i\mathrm{H},\pi^r(x)]=\dot{\pi}^r(x).\end{aligned} \tag{3.9}$$

The Hamiltonian H is obtained by integrating the Hamiltonian density which we defined in (2.29)

$$\mathrm{H}=\int d^3x\,\mathcal{H}(x)=\int d^3x\,\theta^{00}(x)\quad , \tag{3.10}$$

whereas the 3-momentum of the field is given by

$$\mathrm{P}^j=\int d^3x\,\theta^{0j}(x) \tag{3.11}$$

So, let us commute the Hamiltonian density with $\varphi_r(y)$ and $\pi^r(y)$

$$\begin{aligned}&[\mathcal{H}(x),\varphi_r(y)]=[\pi^s(x),\varphi_r(y)]\dot{\varphi}_s(x)+\pi^s(x)[\dot{\varphi}_s(x),\varphi_r(y)]-[\mathcal{L}(x),\varphi_r(y)],\\&[\mathcal{H}(x),\pi^r(y)]=\pi^s(x)[\dot{\varphi}_s(x),\pi^r(y)]-[\mathcal{L}(x),\pi^r(y)].\end{aligned}$$

Inserting (3.6) and (3.7) and using (3.3), we find

$$[i\mathcal{H}(x),\varphi_r(y)]=\delta^3(\mathbf{x}-\mathbf{y})\dot{\varphi}_r(x) \quad , \tag{3.12}$$

$$[i\mathcal{H}(x),\pi^r(y)]=\frac{\partial\mathcal{L}(x)}{\partial\varphi_r}\delta^3(\mathbf{x}-\mathbf{y})+\frac{\partial\mathcal{L}(x)}{\partial(\partial_j\varphi_r)}\partial_j^{(x)}\delta^3(\mathbf{x}-\mathbf{y}) \quad . \tag{3.13}$$

Integrating over x, we find (after integrating (3.13) by part)

$$[i\mathrm{H},\varphi_r(y)]=\dot{\varphi}_r(y),$$
$$[i\mathrm{H},\pi^r(y)]=\frac{\partial\mathcal{L}(y)}{\partial\varphi_r(y)}-\partial_j\left(\frac{\partial\mathcal{L}(y)}{\partial(\partial_j\varphi_r)}\right)=\partial_0\left(\frac{\partial\mathcal{L}(y)}{\partial\dot{\varphi}_r}\right)=\dot{\pi}^r(y),$$

where the Euler-Lagrange equation and the definition of the conjugate momentum (2.28) have been used. In QFT these are called Heisenberg equations which, according to (3.1) and (3.2), correspond to the canonical Hamilton equations in classical field theory

$$\frac{\delta\mathrm{H}}{\delta\pi^r(x)}=\dot{\varphi}_r(x)\,,\quad \frac{\delta\mathrm{H}}{\delta\varphi_r(x)}=-\dot{\pi}(x) \quad . \tag{3.14}$$

In exactly the same way, we can establish the following more general equal-time commutation relations

$$[i\theta^{0\mu}(x),\varphi_r(y)]=\delta^3(\mathbf{x}-\mathbf{y})\,\partial^\mu\varphi_r(x), \tag{3.15}$$

$$[i\theta^{0\mu}(x),\pi^r(y)]=\left(\eta^{0\mu}\frac{\partial\mathcal{L}(x)}{\partial\partial_j\varphi_r}-\eta^{j\mu}\pi^r(x)\right)\partial_j\delta^3(\mathbf{x}-\mathbf{y})+\eta^{0\mu}\frac{\partial\mathcal{L}(x)}{\partial\varphi_r}\delta^3(\mathbf{x}-\mathbf{y}) \tag{3.16}$$

Integrating these over x and introducing the Euler derivative (2.13) in (3.16), we find

$$[i\mathrm{P}^\mu(t),\varphi_r(x)]\doteq[i\int d^3x'\,\theta^{0\mu}(x'),\varphi_r(x)]=\partial^\mu\varphi_r(x) \quad , \tag{3.17}$$

$$[i\mathrm{P}^\mu(t),\pi^r(x)]\doteq[i\int d^3x'\,\theta^{0\mu}(x'),\pi^r(x)]=\partial^\mu\pi^r(x)+\eta^{0\mu}\frac{\hat{\delta}\mathcal{L}}{\hat{\delta}\varphi_r} \quad , \tag{3.18}$$

where

$$\mathrm{P}^\mu(t)=\int d^3x\,\theta^{0\mu}(x) \quad , \tag{3.19}$$

is the field energy-momentum "4-vector". We will talk about the physical meaning of these relations, and show that P^μ is indeed a time-independent 4-vector, when we introduce the Noether charge. We will also see why the Euler-Lagrange equation of motion appears in (3.18) but not in (3.17).

Notice that (3.18) shows the equivalence between Lagrangian and Hamiltonian formalisms

$$\frac{\hat{\delta}\mathcal{L}(x)}{\hat{\delta}\varphi_r(x)}=0 \quad \Leftrightarrow \quad [i\mathrm{P}^\mu\,,\pi^r(x)]=\partial^\mu\pi^r(x)=-\frac{\delta\mathrm{P}^\mu}{\delta\varphi_r(x)} \ . \tag{3.20}$$

Okay, since we calculated the ETCR of the 0-th component of translation current, $\theta^{0\mu}$, with the fields, let us do some justice to the Poincare' group and do the same thing with the 0-th component of Lorentz current $\mathfrak{M}^{o\mu\nu}$. From (2.35) this is given by

$$\mathfrak{M}^{0\nu\sigma}(x)=x^\nu\,\theta^{0\sigma}(x)-x^\sigma\,\theta^{0\nu}(x)-i\pi^r(x)(\Sigma^{\nu\sigma})_r{}^s\,\varphi_s(x) \ . \tag{3.21}$$

Using (3.15) and the fundamental ETCR (3.3), we find

$$[i\mathfrak{M}^{0\nu\sigma}(x),\varphi_l(y)]=\left(x^\nu\partial^\sigma\varphi_l-x^\sigma\partial^\nu\varphi_l-i(\Sigma^{\nu\sigma})_l{}^s\,\varphi_s(x)\right)\delta^3(\mathbf{x}-\mathbf{y}) \ . \tag{3.22}$$

Integrating over x and changing $y\to x$, we get our final result

$$[i\mathrm{M}^{\nu\sigma}(t),\varphi_r(x)]\doteq[i\int d^3x'\,\mathfrak{M}^{0\nu\sigma}(x'),\varphi_r(x)]=x^\nu\partial^\sigma\varphi_r-x^\sigma\partial^\nu\varphi_r-i(\Sigma^{\nu\sigma})_r{}^s\,\varphi_s \ . \tag{3.23}$$

In the next section we will show that the integral

$$\mathrm{M}^{\mu\nu}(t)=\mathrm{L}^{\mu\nu}(t)+\mathrm{S}^{\mu\nu}(t)=\int d^3x\,\mathfrak{M}^{0\mu\nu}(x) \ , \tag{3.24}$$

is a time-independent Lorentz tensor. It represents the total angular momentum of the field, with the orbital angular momentum $\mathrm{L}^{\mu\nu}$ and the spin angular momentum $\mathrm{S}^{\mu\nu}$ are given by

$$\begin{aligned}\mathrm{L}^{\mu\nu}(t)&\doteq\int d^3x\,\mathfrak{L}^{o\mu\nu}(x)=\int d^3x\left(x^\mu\,\theta^{0\nu}(x)-x^\nu\,\theta^{0\mu}(x)\right),\\ \mathrm{S}^{\mu\nu}(t)&\doteq\int d^3x\,\mathcal{S}^{0\mu\nu}(x)=-i\int d^3x\,\pi^r(x)(\Sigma^{\mu\nu})_r{}^s\,\varphi_s(x).\end{aligned} \tag{3.25}$$

Before going any further, let us make the following observations: If we contract (3.17) with the translation parameter a^μ and (3.23) with the Lorentz parameter $\omega^{\nu\sigma}$, we recover the correct infinitesimal Poincare' transformations on the field

$$\delta\varphi_r(x)=[-ia^\mu\,\mathrm{P}_\mu(t),\varphi_r(x)]=-a^\mu\partial_\mu\varphi_r(x) \ , \tag{3.26}$$

$$\begin{aligned}\delta\varphi_r(x)=[\frac{i}{2}\omega^{\nu\sigma}\mathrm{M}_{\nu\sigma}(t),\varphi_r(x)]&=-\omega^\sigma{}_\nu\,x^\nu\,\partial_\sigma\varphi_r-\frac{i}{2}(\omega^{\mu\nu}\Sigma_{\mu\nu})_r{}^s\,\varphi_s(x)\\ &=-\delta x^\sigma\,\partial_\sigma\varphi_r-\frac{i}{2}(\omega\cdot\Sigma)_r{}^s\,\varphi_s(x)\,.\end{aligned} \tag{3.27}$$

Remarkably, the only assumption used to derive these relations is (that the fields satisfy) the fundamental ETCR's (3.3). Indeed, neither dynamical considerations (i.e., equations of motion) nor symmetry considerations (i.e., conservation laws) have been used to obtain (3.26) and (3.27). In fact we will now show that the same result holds for arbitrary group of transformations.

Setting $\mu = 0$ in the Noether current (2.18), we find

$$J^{0}_{(\alpha)}(x) = \pi^{r}(x)\,\delta_{\alpha}\varphi_{r}(x) - \delta_{\alpha}x^{0}\,\mathcal{L}(x) \ . \tag{3.28}$$

Thus, at $x^0 = y^0$

$$[J^{0}_{(\alpha)}(x), \varphi_{s}(y)] = \pi^{r}(x)[\delta_{\alpha}\varphi_{r}(x), \varphi_{s}(y)] + [\pi^{r}(x), \varphi_{s}(y)]\delta_{\alpha}\varphi_{r}(x) - \delta_{\alpha}x^{0}[\mathcal{L}(x), \varphi_{s}(y)] \ .$$

Due to (1.15) and (3.6) the first and the third terms add up to zero, and the second term, when evaluated from the ETCR (3.3), gives

$$[iJ^{0}_{(\alpha)}(x), \varphi_{s}(y)] = \delta_{\alpha}\varphi_{s}(x)\,\delta^{3}(\mathbf{x}-\mathbf{y}) \ . \tag{3.29}$$

Hence, a 3-volume integration over x will generate the infinitesimal transformations on the fields

$$\delta_{\alpha}\varphi_{r}(x) = [iQ_{\alpha}(t), \varphi_{r}(x)] \ , \tag{3.30}$$

where $Q_{\alpha}(t)$ (the Noether charge) is given by the integral

$$Q_{\alpha}(t) = \int d^{3}x\, J^{0}_{(\alpha)}(x) \ . \tag{3.31}$$

Again, the important fact about (3.30) is that it has been derived for arbitrary group of transformations and without reference to a specific form of $\mathcal{L}(x)$, i.e., without any commitment to symmetry or dynamics. Therefore, even when the transformation is not a symmetry operation and at least formally, it seems that (3.30) and its local version (3.29) are always valid. Thus, any result that can be derived from them will necessarily be true if we ignore the usual difficulties of local quantum field theory.

The so-called Ward-Takahashi identity in QFT is just an alternative form of the canonical equal-time relation (3.30). It can be derived from the following time-ordered product

$$\mathrm{T}\left(J^{\mu}_{(\alpha)}(x)\,\varphi_{r}(y)\right) = \Theta(x^{0}-y^{0})J^{\mu}_{(\alpha)}(x)\,\varphi_{r}(y) + \Theta(y^{0}-x^{0})\varphi_{r}(y)\,J^{\mu}_{(\alpha)}(x) \ . \tag{3.32}$$

Differentiating this with respect to x and using the relations

$$\begin{aligned} \partial^{(x)}_{\mu}\Theta(x^{0}-y^{0}) &= \eta_{\mu 0}\,\delta(x^{0}-y^{0}), \\ \partial^{(x)}_{\mu}\Theta(y^{0}-x^{0}) &= -\eta_{\mu 0}\,\delta(x^{0}-y^{0}), \end{aligned} \tag{3.33}$$

we find

$$\partial_\mu \mathrm{T}\left(J^\mu_{(\alpha)}(x)\varphi_r(y)\right) = \mathrm{T}\left(\partial_\mu J^\mu_{(\alpha)}(x)\,\varphi_r(y)\right) + \delta(x^0 - y^0)[J^0_{(\alpha)}(x),\varphi_r(y)] \ . \quad (3.34)$$

The presence of the delta function allows us to use (3.29) in the second term on the right and get

$$\partial_\mu \mathrm{T}\left(J^\mu_{(\alpha)}(x)\varphi_r(y)\right) = \mathrm{T}\left(\partial_\mu J^\mu_{(\alpha)}\,\varphi_r(y)\right) - i\delta^4(x-y)\,\delta_{(\alpha)}\varphi_r(x) \ . \quad (3.35)$$

Now, on integrating (3.35) over a space-time region Ω containing the point y, we arrive at the Ward-Takahashi identity

$$i\delta_\alpha\varphi_r(y) = \int_\Omega d^4x\,\mathrm{T}\left(\partial_\mu J^\mu_{(\alpha)}(x)\varphi_r(y)\right) - \int_\Omega d^4x\,\partial_\mu \mathrm{T}\left(J^\mu_{(\alpha)}(x)\varphi_r(y)\right) \ . \quad (3.36)$$

Since the variation δ_α does not upset the time ordering,

$$i\delta_\alpha \mathrm{T}\left(\varphi_{r_1}(y_1)\varphi_{r_2}(y_2)\cdots\varphi_{r_n}(y_n)\right) = i\sum_{i=1}^{n}\mathrm{T}\left(\varphi_{r_1}(y_1)\cdots\delta_\alpha\varphi_{r_i}(y_i)\cdots\varphi_{r_n}(y_n)\right) \ , \quad (3.37)$$

we can generalize (3.36) and obtain, for all $y_i \in \Omega$, a general form for the W-T identity

$$i\sum_{i=1}\mathrm{T}\left(\varphi_{r_1}(y_1)\cdots\delta_\alpha\varphi_{r_i}(y_i)\cdots\right) = \int_\Omega d^4x\,\mathrm{T}\left(\partial_\mu J^\mu_\alpha(x)\,\varphi_{r_1}(y_1)\cdots\right) - \int_\Omega d^4x\,\partial_\mu \mathrm{T}\left(J^\mu_\alpha(x)\varphi_{r_1}(y_1)\cdots\right)$$

(3.38)

It is clear that the first term on the right hand side of (3.36) and (3.38) vanishes if the current is conserved, i.e., when the transformation is a symmetry transformation.

Let us now go back to (3.25) and use it to build up the equal time commutator $[i\mathrm{M}^{\mu\nu},\pi^r(x)]$. For the spin angular momentum, we find

$$[i\mathrm{S}^{\mu\nu},\pi^r(x)] = \int d^3x'\,\pi^l(x')(\Sigma^{\mu\nu})_l^{\ s}\,[\varphi_s(x'),\pi^r(x)] \ .$$

Using (3.3), this becomes

$$[i\mathrm{S}^{\mu\nu},\pi^r(x)] = i\pi^s(x)(\Sigma^{\mu\nu})_s^{\ r} \ . \quad (3.39)$$

To calculate the commutator with the orbital angular momentum, we multiply (3.16) by (ix^μ) and integrate over x. Integration by parts then gives

$$[i\int d^3x\,x^\mu\,\theta^{0\nu}(x),\pi^r(y)] = \eta^{\nu j}\partial_j\left(y^\mu\pi^r(y)\right) - \partial_j\left(y^\mu\eta^{0\nu}\frac{\partial\mathcal{L}(y)}{\partial(\partial_j\varphi_r)}\right) + y^\mu\eta^{0\nu}\frac{\partial\mathcal{L}(y)}{\partial\varphi_r}$$

(3.40)

Using the identities

$$\begin{aligned} \partial^{\nu} - \eta^{\nu 0}\,\partial_0 &= \eta^{\nu j}\,\partial_j\,, \\ \eta^{\mu\nu} - \delta^{\mu}_{0}\,\eta^{0\nu} &= \delta^{\mu}_{j}\,\eta^{j\nu}\,, \end{aligned} \tag{3.41}$$

in (3.38), we find after some rearrangement

$$\begin{aligned} [i\int d^3x x^{\mu}\theta^{0\nu}, \pi^r(y)] = \eta^{\mu\nu}\pi^r + y^{\mu}\partial^{\nu}\pi^r - \eta^{0\nu}\left(\delta^{\mu}_{0}\,\pi^r + \delta^{\mu}_{j}\,\frac{\partial\mathcal{L}}{\partial(\partial_j\varphi_r)}\right) \\ + y^{\mu}\eta^{0\nu}\left(\frac{\partial\mathcal{L}}{\partial\varphi_r} - \dot{\pi}^r - \partial_j(\frac{\partial\mathcal{L}}{\partial(\partial_j\varphi_r)})\right). \end{aligned}$$

In this equation, we use the definition of the conjugate momentum

$$\pi^r(y) = \frac{\partial\mathcal{L}(y)}{\partial(\partial_0\varphi_r)}\;,\quad \dot{\pi}^r(y) = \partial_0\left(\frac{\partial\mathcal{L}(y)}{\partial(\partial_0\varphi_r)}\right)\;, \tag{3.42}$$

to obtain

$$[i\int d^3x\, x^{\mu}\,\theta^{0\nu}, \pi^r(y)] = \eta^{\mu\nu}\pi^r + y^{\mu}\partial^{\nu}\pi^r - \delta^{\mu}_{\sigma}\eta^{\nu 0}\,\frac{\partial\mathcal{L}}{\partial(\partial_{\sigma}\varphi_r)} + y^{\mu}\eta^{\nu 0}\,\frac{\hat{\delta}\mathcal{L}}{\hat{\delta}\varphi_r}\;. \tag{3.43}$$

Thus, the required commutator follows by the following anti-symmetric combination

$$[i\mathrm{L}^{\mu\nu}, \pi^r(x)] = x^{[\mu}\partial^{\nu]}\,\pi^r - \delta^{[\mu}_{\sigma}\eta^{\nu]0}\,\frac{\partial\mathcal{L}(x)}{\partial(\partial_{\sigma}\varphi_r)} + x^{[\mu}\eta^{\nu]0}\,\frac{\hat{\delta}\mathcal{L}(x)}{\hat{\delta}\varphi_r}\;. \tag{3.44}$$

Adding (3.42) to (3.37), we get

$$[i\mathrm{M}^{\mu\nu}, \pi^r(x)] = x^{\mu}\partial^{\nu}\pi^r(x) - x^{\nu}\partial^{\mu}\pi^r(x) + i\pi^s(x)(\Sigma^{\mu\nu})_s{}^r + \mathrm{F}^{\mu\nu r}\;. \tag{3.45}$$

The extra term in the Lorentz transformation law for π^r is given by

$$\mathrm{F}^{\mu\nu r} = -\mathrm{F}^{\nu\mu r} \doteq x^{[\mu}\eta^{\nu]0}\,\frac{\hat{\delta}\mathcal{L}(x)}{\hat{\delta}\varphi_r} - \delta^{[\mu}_{\sigma}\eta^{\nu]0}\,\frac{\partial\mathcal{L}(x)}{\partial(\partial_{\sigma}\varphi_r)}\;. \tag{3.46}$$

This extra piece does not vanish even when the equations of motion are satisfied. This is because the set $\{\pi^r(x)\}$ does not form a covariant manifold under the Poincare' group.

Now, let us use Heisenberg equations (3.9) to prove the following theorem

Theorem 3.1

If the equal-time commutation relations (3.3) are valid at a certain time $x^0 = y^0 = t$ *, they are also valid at* $t+\varepsilon$ *.*

Proof:

The proof will be based on Heisenberg equations and the following Jacobi identity

$$[\varphi_r(\mathbf{x},t),[i\mathrm{H},\pi^s(\mathbf{y},t)]]+[\pi^s(\mathbf{y},t),[\varphi_r(\mathbf{x},t),i\mathrm{H}]]+[i\mathrm{H},[\pi^s(\mathbf{y},t),\varphi_r(\mathbf{x},t)]]=0 \ .$$

Since at $x^0 = y^0 = t$, we have

$$[\varphi_r(\mathbf{x},t),\pi^s(\mathbf{y},t)]=i\delta_r^s\,\delta^3(\mathbf{x}-\mathbf{y}) \ .$$

Thus the third term in the Jacobi identity vanishes because the deltas are c-numbers. And we are left with

$$[\varphi_r(\mathbf{x},t),[i\mathrm{H},\pi^s(\mathbf{y},t)]]+[[i\mathrm{H},\varphi_r(\mathbf{x},t)],\pi^s(\mathbf{y},t)]=0 \ . \tag{3.47}$$

Using Heisenberg equations (3.9) in (3.45), we find

$$[\varphi_r(\mathbf{x},t),\dot{\pi}^s(\mathbf{y},t)]+[\dot{\varphi}_r(\mathbf{x},t),\pi^s(\mathbf{y},t)]=0 \tag{3.48}$$

Multiplying this by ε and adding the result to the fundamental equal-time commutation relation, we find

$$[\varphi_r(\mathbf{x},t),\pi^s(\mathbf{y},t)]+[\varphi_r(\mathbf{x},t),\varepsilon\dot{\pi}^s(\mathbf{y},t)]+[\varepsilon\dot{\varphi}_r(\mathbf{x},t),\pi^s(\mathbf{y},t)]=i\delta_r^s\,\delta^3(\mathbf{x}-\mathbf{y})$$
(3.49)

Thus, to first order in ε we can write this as

$$\Big[\big(\varphi_r(\mathbf{x},t)+\varepsilon\dot{\varphi}_r(\mathbf{x},t)\big),\big(\pi^s(\mathbf{y},t)+\varepsilon\dot{\pi}^s(\mathbf{y},t)\big)\Big]=i\delta_r^s\,\delta^3(\mathbf{x}-\mathbf{y}) \ , \tag{3.50}$$

or

$$[\varphi_r(\mathbf{x},t+\varepsilon),\pi^s(\mathbf{y},t+\varepsilon)]=i\delta_r^s\,\delta^3(\mathbf{x}-\mathbf{y}) \ . \tag{3.51}$$

And, by exactly the same method, we can establish the remaining commutation relations

$$[\varphi_r(\mathbf{x},t+\varepsilon),\varphi_s(\mathbf{y},t+\varepsilon)]=[\pi^r(\mathbf{x},t+\varepsilon),\pi^s(\mathbf{y},t+\varepsilon)]=0 \ . \tag{3.52}$$

qed

References and Further Reading

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