In the general case, deforming the spring from position x_1 to x_2,

$$U = \frac{1}{2}k\left(x_2^2 - x_1^2\right)$$

Notation for Potential Energy

The change in the potential energy V of a system is

$$U = -\Delta V$$

Note that negative work is done by a system while its own potential energy is increased by the action of an external force or moment. The external agent does positive work at the same time since it acts in the same direction as the resulting displacement.

Potential Energy at Equilibrium

For equilibrium of a system,

$$\frac{dV}{dq} = 0$$

where q = an independent coordinate along which there is possibility of displacement.

For a system with n degrees of freedom,

$$\frac{\partial V}{\partial q_i} = 0, \qquad i = 1, 2, \ldots, n$$

Equilibrium is stable if $(d^2V/dq^2) > 0$.

Equilibrium is unstable if $(d^2V/dq^2) < 0$.

Equilibrium is neutral only if all derivatives of V are zero. In cases of complex configurations, evaluate derivatives of higher order as well.

Moments of Inertia

The topics of inertia are related to the methods of first moments. They are traditionally presented in statics in preparation for application in dynamics or mechanics of materials.

Moments of Inertia of a Mass

The moment of inertia dI_x of an elemental mass dM about the x axis (Figure 1.2.28) is defined as

$$dI_x = r^2\ dM = \left(y^2 + z^2\right) dM$$

where r is the nearest distance from dM to the x axis. The moments of inertia of a body about the three coordinate axes are

$$I_x = \int r^2\ dM = \int \left(y^2 + z^2\right) dM$$

$$I_y = \int \left(x^2 + z^2\right) dM \tag{1.2.34}$$

$$I_z = \int \left(x^2 + y^2\right) dM$$

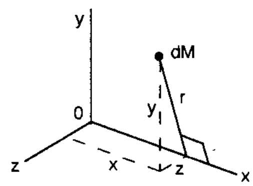

FIGURE 1.2.28 Mass element dM in xyz coordinates.

Radius of Gyration. The radius of gyration r_g is defined by $r_g = \sqrt{I_x / M}$, and similarly for any other axis. It is based on the concept of the body of mass M being replaced by a point mass M (same mass) at a distance r_g from a given axis. A thin strip or shell with all mass essentially at a constant distance r_g from the axis of reference is equivalent to a point mass for some analyses.

Moment of Inertia of an Area

The moment of inertia of an elemental area dA about the x axis (Figure 1.2.29) is defined as

$$dI_x = y^2\ dA$$

where y is the nearest distance from dA to the x axis. The moments of inertia (second moments) of the area A about the x and y axes (because A is in the xy plane) are

$$I_x = \int y^2\ dA \qquad I_y = \int x^2\ dA \tag{1.2.35}$$

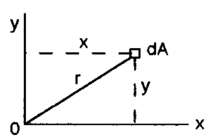

FIGURE 1.2.29 Area A in the xy plane.

The radius of gyration of an area is defined the same way as it is for a mass: $r_g = \sqrt{I_x / A}$, etc.

Polar Moment of Inertia of an Area

The polar moment of inertia is defined with respect to an axis perpendicular to the area considered. In Figure 1.2.29 this may be the z axis. The polar moment of inertia in this case is

$$J_O = \int r^2\ dA = \int \left(x^2 + y^2\right) dA = I_x + I_y \tag{1.2.36}$$

Parallel-Axis Transformations of Moments of Inertia

It is often convenient to first calculate the moment of inertia about a centroidal axis and then transform this with respect to a parallel axis. The formulas for the transformations are

$$I = I_C + Md^2 \qquad \text{for a mass } M$$

$$I = I_C + Ad^2 \qquad \text{for an area } A \tag{1.2.37}$$

$$J_O = J_C + Ad^2 \qquad \text{for an area } A$$

where I or J_O = moment of inertia of M or A about any line ℓ
I_C or J_C = moment of inertia of M or A about a line through the mass center or centroid and parallel to ℓ
d = nearest distance between the parallel lines

Note that one of the two axes in each equation must be a centroidal axis.

Products of Inertia

The products of inertia for areas and masses and the corresponding parallel-axis formulas are defined in similar patterns. Using notations in accordance with the preceding formulas, products of inertia are

$$I_{xy} = \int xy\, dA \quad \text{for area,} \qquad \text{or} \quad \int xy\, dM \quad \text{for mass}$$

$$I_{yz} = \int yz\, dA \qquad \text{or} \quad \int yz\, dM \tag{1.2.38}$$

$$I_{xz} = \int xz\, dA \qquad \text{or} \quad \int xz\, dM$$

Parallel-axis formulas are

$$I_{xy} = I_{x'y'} + A\, d_x d_y \quad \text{for area,} \qquad \text{or} \quad I_{x'y'} + M\, d_x d_y \quad \text{for mass}$$

$$I_{yz} = I_{y'z'} + A\, d_y d_z \qquad \text{or} \quad I_{y'z'} + M\, d_y d_z \tag{1.2.39}$$

$$I_{xz} = I_{x'z'} + A\, d_x d_z \qquad \text{or} \quad I_{x'z'} + M\, d_x d_z$$

Notes: The moment of inertia is always positive. The product of inertia may be positive, negative, or zero; it is zero if x or y (or both) is an axis of symmetry of the area. Transformations of known moments and product of inertia to axes that are inclined to the original set of axes are possible but not covered here. These transformations are useful for determining the principal (maximum and minimum) moments of inertia and the principal axes when the area or body has no symmetry. The principal moments of inertia for objects of simple shape are available in many texts.