Chapter 3

Some QED Vacuum Effects

> [My father] said, "I understand that they say that light is emitted from an atom when it goes from one state to another, from an excited state to a state of lower energy."
>
> I said, "That's right."
>
> "And light is a kind of particle, a photon, I think they call it."
>
> "Yes."
>
> "So if the photon comes out of the atom when it goes from the excited to the lower state, the photon must have been in the atom in the excited state."
>
> I said, "Well, no."
>
> He said, "Well, how do you look at it so you can think of a particle photon coming out without it having been there in the excited state?"
>
> I thought a few minutes, and I said, "I'm sorry; I don't know. I can't explain it to you."
>
> — Richard P. Feynman, *The Physics Teacher* (September 1969).

3.1 Introduction

We noted in the preceding chapter that Dirac's theory of emission and absorption (1927) was the first application of the quantum theory of radiation. The importance of Dirac's theory of spontaneous emission has been emphasized by Weinberg (1977):

> ... This problem was of crucial importance, because the process of spontaneous emission of radiation is one in which "particles" are

> actually created. Before the event, the system consists of an excited atom, whereas after the event, it consists of an atom in a state of lower energy, plus one photon. If quantum mechanics could not deal with processes of creation and destruction, it could not be an all-embracing physical theory ... Dirac's successful theory of spontaneous emission of radiation confirmed the universal character of quantum mechanics.

Dirac argued that his theory "must presumably give the effect of radiation reaction on the emitting system." Spontaneous emission was also interpreted in terms of radiation reaction in the theory of Landau (1927) and, before the development of the quantum formalism, by van Vleck (1924). However, contemporary physicists, when asked to give a physical explanation for the occurrence of spontaneous emission, generally invoke the vacuum electromagnetic field. This view was popularized by Weisskopf (1935) and later by Welton (1948), who argued that spontaneous emission "can be thought of as forced emission taking place under the action of the fluctuating field." In the following chapters we show that these two interpretations — based on radiation reaction or vacuum field fluctuations — are in fact closely related in the quantum theory of radiation. We show furthermore that various other effects can be interpreted equally well in terms of radiation reaction or vacuum field fluctuations.

What are these "vacuum fluctuation effects"? The first example that is usually cited is the Lamb shift, or sometimes the Casimir force between conducting plates. In this chapter we consider these and other manifestations of the vacuum electromagnetic field. Our aim is not to present detailed calculations, but to emphasize the *physics* of the vacuum field. For this reason we adhere strictly to the formalism of the quantized field only when it is absolutely necessary.

3.2 Spontaneous Emission

Spontaneous emission is ultimately responsible for most of the light around us. For a thermal source the ratio of the spontaneous and stimulated emission rates for radiation of frequency ω_o is (Section 1.8)

$$\frac{A_{21}}{B_{21}\rho(\omega_o)} = e^{\hbar\omega_o/kT} - 1. \tag{3.1}$$

The sun may be regarded for our purposes as a blackbody radiator at the temperature $T = 6000$ K. At this temperature the ratio (3.1) is about 400 at the wavelength $\lambda = 400$ nm, and about 30 at $\lambda = 700$ nm. Most of the visible output from the sun, therefore, is due to spontaneous rather than stimulated emission.

As we shall see in the following chapter, spontaneous emission can be correctly described only when the radiation field is quantized; if the field is not treated quantum mechanically we obtain predictions in conflict with experiment. Nevertheless some aspects of spontaneous emission are adequately described without the full machinery of quantum electrodynamics. In this section we use some simplistic arguments to derive the A coefficient for spontaneous emission and to provide some preliminary evidence for the interplay between radiation reaction and vacuum field fluctuations.

Consider the rate R_{21} of stimulated emission in a broadband field of spectral energy density $\rho(\omega_o)$. According to the discussion in Section 1.8, this rate is $B_{21}\rho(\omega_o)$ for an atomic transition from level 2 to level 1 with transition frequency $\omega_o = (E_2 - E_1)/\hbar$. The Einstein B coefficient for stimulated emission is given by the standard formula

$$B_{21} = \frac{4\pi^2 d^2}{3\hbar^2}\ , \tag{3.2}$$

where $\mathbf{d}$ is the electric dipole matrix element for the transition $2 \rightarrow 1$ and $d \equiv |\mathbf{d}|$. Thus

$$R_{21} = \frac{4\pi^2 d^2}{3\hbar^2}\rho(\omega_o). \tag{3.3}$$

This result follows also from the classical formula (1.7) when we replace e^2/m by $(e^2/m)f$, with f the transition oscillator strength defined as $2md^2\omega_o/e^2\hbar$. (See the remark at the end of Appendix A.) That is,

$$\frac{\dot{W}_A}{\hbar\omega_o} = \frac{2\pi^2 e^2}{3m\hbar\omega_o}\rho(\omega_o) \rightarrow \frac{2\pi^2 e^2}{3m\hbar\omega_o} f\rho(\omega_o) = R_{21}. \tag{3.4}$$

According to equation (3.3) the vacuum electromagnetic field should induce an atom in the excited level 2 to make a downward transition to level 1 at the rate (transition probability per unit time)

$$R_{\mathrm{VF}} = \left(\frac{4\pi^2 d^2}{3\hbar^2}\right)\left(\frac{\hbar\omega_o^3}{2\pi^2 c^3}\right) = \frac{2d^2\omega_o^3}{3\hbar c^3} = \frac{1}{2}A_{21}, \tag{3.5}$$

where A_{21} is the Einstein A coefficient for spontaneous emission (Chapter 4).

We have thus arrived at the same result found in Section 1.9: the vacuum field induces transitions at a rate equal to *half* the spontaneous emission rate. Evidently spontaneous emission *cannot* simply "be thought of as forced emission taking place under the action of the fluctuating field."

Consider now the effect of radiation reaction. As shown in Appendix A, the radiation reaction field is responsible for the rate $2e^2a^2/3c^3$ at which an

oscillating charge loses energy to the electromagnetic field. For oscillation at frequency ω_o with amplitude x_o this rate is

$$\frac{d\epsilon}{dt} = \frac{2e^2\omega_o^4 x_o^2}{3c^3}\cos^2\omega_o t \rightarrow \frac{e^2\omega_o^4 x_o^2}{3c^3} \tag{3.6}$$

when we average over a cycle of oscillation. Since $\epsilon = m\omega_o^2 x_o^2$ is the energy of oscillation, we have

$$\frac{d\epsilon}{dt} = \left(\frac{e^2\omega_o^2}{3mc^3}\right)\epsilon, \tag{3.7}$$

and so $e^2\omega_o^2/3mc^3$ is the rate of emission attributable to radiation reaction. The replacement of e^2/m by $e^2 f/m$ as previously gives

$$R_{\mathrm{RR}} = \left(\frac{e^2\omega_o^2}{3mc^3}\right)\left(\frac{2md^2\omega_o}{e^2\hbar}\right) = \frac{2d^2\omega_o^3}{3\hbar c^3} = \frac{1}{2}A_{21} \tag{3.8}$$

for the emission rate due to radiation reaction.

On the basis of this simplistic semiclassical analysis, therefore, we have arrived at the conclusion that $R_{\mathrm{RR}} = R_{\mathrm{VF}} = \frac{1}{2}A_{21}$ and

$$A_{21} = R_{\mathrm{VF}} + R_{\mathrm{RR}}. \tag{3.9}$$

In other words, both the vacuum field and radiation reaction induce transitions at the rate $\frac{1}{2}A_{21}$, and the two together give the Einstein A coefficient for spontaneous emission.

As noted earlier, modern physicists generally think of spontaneous emission as a consequence of the vacuum field. Weisskopf (1981), for instance, writes that "spontaneous emission appears as a forced emission caused by the zero-point oscillations of the electromagnetic field." The fact that the vacuum field gives only half the correct A coefficient in this simplified picture does not seem to be widely appreciated, although it has been emphasized by several authors (Ginzburg, 1983; Milonni, 1984), and in his well-known textbook Schiff (1968) indirectly acknowledges it:

> ... From a formal point of view, we can say that the spontaneous emission probability is equal to the probability of emission that would be induced by the presence of one quantum in each state of the radiation field. Now ... the smallest possible energy of the field corresponds to the presence of one-half quantum per state. This suggests that we regard the spontaneous emission as being induced by the zero-point oscillations of the electromagnetic field; note, however, that these oscillations are twice as effective in producing emissive transitions as are real photons and are of course incapable of producing absorptive transitions.

These arguments indicate that the "missing one-half" comes from radiation reaction. Of course these arguments are semiclassical and oversimplified; the main purpose of the following chapter is to refine these arguments using the quantum theory of radiation. We will also show why the vacuum field is "of course incapable of producing absorptive transitions": in the lower state of an atomic transition the effects of the vacuum field and radiation reaction cancel, so that the "spontaneous absorption" rate is

$$A_{12} = R_{\rm VF} - R_{\rm RR} = \frac{1}{2}A_{21} - \frac{1}{2}A_{21} = 0. \tag{3.10}$$

3.3 Atomic Stability

The fact that an accelerating charge loses energy by radiating implies, according to classical ideas, that an electron should spiral into the nucleus and that atoms should not be stable. The balancing of the effects of radiation reaction and the vacuum field implied by (3.10), however, suggests that the stability of atoms might be attributable to the influence on the atom of the vacuum field. We now give a simplistic argument in support of this idea.

Using equation (3.4) we write

$$\dot{W}_{\rm A} = \frac{2\pi^2 e^2}{3m} f\rho(\omega_o) = \frac{e^2 f\hbar\omega_o^3}{3mc^3} \tag{3.11}$$

for the rate at which an atom absorbs energy from the vacuum field. But according to (3.6) there is also a loss of energy at the rate

$$\dot{W}_{\rm EM} = \frac{e^2 f\omega_o^4 x_o^2}{3c^3} \tag{3.12}$$

due to radiation, where we have again made the replacement $e^2/m \rightarrow e^2 f/m$. Equating (3.11) and (3.12), we obtain

$$m x_o^2 \omega_o = \hbar, \tag{3.13}$$

which will be recognized as the Bohr quantization condition for the ground state of a one-electron atom.

This "derivation" of the Bohr quantization condition obviously should not be taken very seriously. It suggests only how Bohr's quantization condition, at least for $n = 1$, *might* have been interpreted by physicists in 1913. We now know that the vacuum field is in fact formally necessary for the stability of atoms in quantum theory: as we saw in Section 2.6, radiation reaction will cause canonical commutators like $[x, p_x]$ to decay to zero unless the fluctuating vacuum field is included, in which case commutators are consistently preserved.

3.4 The Lamb Shift

The solution of the Schrödinger equation for the hydrogen atom gives energy levels depending only on the principal quantum number n. In the solution of the Dirac equation (Chapter 9) the spin–orbit coupling partially lifts this degeneracy, but states with the same n and the same total angular momentum quantum number j, such as $2s_{1/2}$ and $2p_{1/2}$, remain degenerate:

$$E = mc^2 \left[1 + \left(\frac{\alpha}{n - (j + \frac{1}{2}) + \sqrt{(j + \frac{1}{2})^2 - \alpha^2}} \right)^2 \right]^{-1/2} , \qquad (3.14)$$

where $\alpha = e^2/\hbar c \cong 1/137$ is the fine structure constant, $n = 1, 2, 3, ...\infty$, and $j + \frac{1}{2} \leq n$. Experiments in the 1930s indicated that the $2s_{1/2}$ and $2p_{1/2}$ energies might actually differ, but the data were not sufficiently accurate to draw any definite conclusions, and other experiments appeared to confirm the prediction of degeneracy. In 1947, however, Lamb and Retherford performed experiments showing convincingly that the $2s_{1/2}$ level lies about 1000 MHz, or 0.030 cm^{-1}, above the $2p_{1/2}$ level. Shortly thereafter they reported a more accurate value near 1060 MHz. This tiny energy difference is called the Lamb shift.

According to the energy level formula (3.14) predicted by the Dirac equation, the energy difference between the $2p_{3/2}$ and $2p_{1/2}, 2s_{1/2}$ levels is $\cong \alpha^4 mc^2/32$, corresponding to a frequency of about 11,000 MHz or a wavelength of about 2.7 cm. A simplified energy level diagram for the $n = 2$ states of hydrogen, including the Lamb shift, is shown in Figure 3.1.

The fact that the $2s_{1/2} - 2p_{3/2}$ (and $2s_{1/2} - 2p_{1/2}$) transition wavelength lies in the microwave region allowed Lamb and Retherford to utilize advances in microwave technology made during World War II. The basic idea of their experiment is as follows. First a beam of H atoms is produced by thermal dissociation of H_2 in an oven. The atomic beam is then bombarded with an electron beam that collisionally excites about 10^{-8} of the atoms into the $2s_{1/2}$ state. This state is metastable, since (one-photon) spontaneous emission to the $1s_{1/2}$ ground state is forbidden ($\Delta l = 0$). The radiative lifetime of the $2s_{1/2}$ state is thus very large ($\cong 1/7$ sec) and is due to two-photon spontaneous emission to the ground state. The $2s_{1/2}$ atoms are detected by the fact that they cause emission of electrons when they are incident on a metal target. Excited atoms incident on the metal thus produce an electric current, while ground-state atoms do not. Now the application of a field at the $2s_{1/2} - 2p_{3/2}$ (or $2s_{1/2} - 2p_{1/2}$) transition frequency induces transitions to a p state which quickly decays to the ground

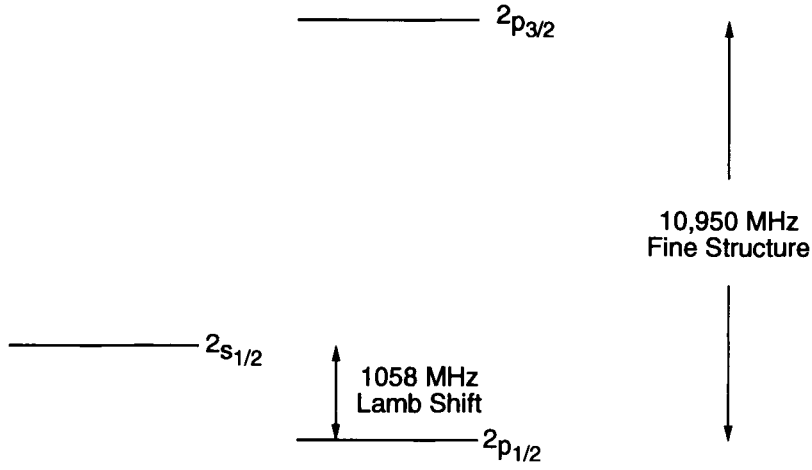

Figure 3.1: Energy level diagram for the $n = 2$ states of the hydrogen atom.

state by spontaneous emission. Application of a field of the appropriate frequency thus reduces the electric current at the detector of excited atoms. (The microwave field can be held fixed while the transition frequency is Zeeman-shifted with a magnetic field.) In this way Lamb and Retherford (1947) determined that the $2s_{1/2} - 2p_{1/2}$ level shift was about 1000 MHz. In 1952 they reported a value of 1058.27 ± 1.0 MHz based on more refined measurements. The Lamb–Retherford experiments and analysis were remarkably accurate. Since then various other experimental techniques have been employed, and the currently accepted value for the $2s_{1/2} - 2p_{1/2}$ shift in hydrogen is about 1057.85 MHz.[1]

The Lamb shift and its explanation marked the beginning of modern quantum electrodynamics. In the words of Dirac (1989), "No progress was made for 20 years. Then a development came, initiated by Lamb's discovery and explanation of the Lamb shift, which fundamentally changed the character of theoretical physics. It involved setting up rules for discarding ... infinities ... "

The reason the Dirac theory leading to (3.14) fails to account for the Lamb shift is that it ignores the coupling of the atomic electron to the vacuum electromagnetic field. Actually the Lamb shift turns out to be a predominantly nonrelativistic effect, and can be understood in part by modifying the Schrödinger theory of the hydrogen atom to include the cou-

[1] For a review see Drake (1982).

pling to the vacuum field. We therefore consider the nonrelativistic theory with Hamiltonian (Chapter 4)

$$H = H_{\rm A} + H_{\rm F} - \frac{e}{mc}\mathbf{A}\cdot\mathbf{p} + \frac{e^2}{2mc^2}\mathbf{A}^2, \tag{3.15}$$

where $H_{\rm A}$ is the Hamiltonian operator for the atomic electron and, as in the preceding chapter, $\mathbf{A}$ is the vector potential, $H_{\rm F}$ is the field Hamiltonian, and we make the electric dipole approximation of neglecting any spatial variation of $\mathbf{A}$.

If the field is treated according to standard classical electromagnetic theory, the vector potential $\mathbf{A} = 0$ in the vacuum and so there is no field to perturb the atomic energy levels. This is not the case when the field is quantized; standard second-order perturbation theory gives the following expression for the shift in the atomic level n due to the interaction $-(e/mc)\mathbf{A}\cdot\mathbf{p}$:

$$\Delta E_n = \sum_m \sum_{\mathbf{k}\lambda} \frac{|\langle m, 1_{\mathbf{k}\lambda}|h_{\mathbf{k}\lambda}|n, \text{vac}\rangle|^2}{E_n - E_m - \hbar\omega_k}, \tag{3.16}$$

$$h_{\mathbf{k}\lambda} = -\frac{e}{mc}\left(\frac{2\pi\hbar c^2}{\omega_k V}\right)^{1/2} a^\dagger_{\mathbf{k}\lambda}(\mathbf{e}_{\mathbf{k}\lambda}\cdot\mathbf{p}), \tag{3.17}$$

where we follow the notation of Chapter 2 for the field in free space. Hence, ΔE_n is the energy shift of the state $|n, \text{vac}\rangle$ in which the atom is in stationary state n and the field is in its vacuum state of no photons. The intermediate state $|m, 1_{\mathbf{k}\lambda}\rangle$ corresponds to the atom in state m and one photon in mode $(\mathbf{k}, \lambda)$. This intermediate state has energy $E_m + \hbar\omega_k$ which appears in the denominator in (3.16). Only one-photon intermediate states appear because $\mathbf{A}$ can only connect the vacuum state to such states , and furthermore only $a^\dagger_{\mathbf{k}\lambda}$ in $\mathbf{A}$ contributes to the matrix element in (3.16) because $a_{\mathbf{k}\lambda}|\text{vac}\rangle = 0$. The expression (3.16) for ΔE_n is derived in the following chapter in both the Schrödinger and Heisenberg pictures.

Since

$$\langle m, 1_{\mathbf{k}\lambda}|h_{\mathbf{k}\lambda}|n, \text{vac}\rangle = -\frac{e}{mc}\left(\frac{2\pi\hbar c^2}{\omega_k V}\right)^{1/2} \mathbf{p}_{mn}\cdot\mathbf{e}_{\mathbf{k}\lambda}, \tag{3.18}$$

we can write

$$\Delta E_n = \frac{2\pi e^2}{m^2}\frac{1}{V}\sum_m \sum_{\mathbf{k}\lambda} \frac{1}{\omega_k}\frac{|\mathbf{p}_{mn}\cdot\mathbf{e}_{\mathbf{k}\lambda}|^2}{\omega_{nm} - \omega_k}, \tag{3.19}$$

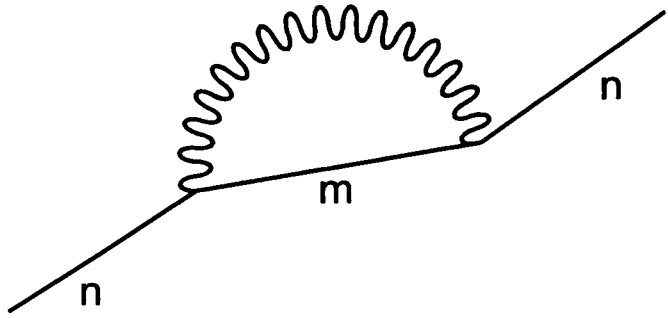

Figure 3.2: Diagrammatic representation of (3.21) in terms of emission and absorption of virtual photons.

where $\hbar\omega_{nm} = E_n - E_m$. Using (2.71) and other simple manipulations, furthermore, we have

$$\begin{aligned}\Delta E_n &= \frac{2e^2}{3\pi m^2 c^3}\sum_m |\mathbf{p}_{mn}|^2 \int_0^\infty \frac{d\omega\omega}{\omega_{nm}-\omega} \\ &= \frac{2\alpha}{3\pi}\left(\frac{1}{mc}\right)^2 \sum_m |\mathbf{p}_{mn}|^2 \int_0^\infty \frac{dEE}{E_n - E_m - E}\,. \end{aligned} \tag{3.20}$$

In these expressions the integrals are to be understood in terms of the Cauchy principal part.

At the risk of laboring the obvious, we emphasize that ΔE_n arises from the vacuum field. Writing

$$\begin{aligned}|\langle m, 1_{\mathbf{k}\lambda}|a^\dagger_{\mathbf{k}\lambda}(\mathbf{p}\cdot\mathbf{e}_{\mathbf{k}\lambda})|n,\text{vac}\rangle|^2 &= \langle n,\text{vac}|a_{\mathbf{k}\lambda}(\mathbf{p}\cdot\mathbf{e}_{\mathbf{k}\lambda})|m,1_{\mathbf{k}\lambda}\rangle \\ &\quad\times\langle m,1_{\mathbf{k}\lambda}|a^\dagger_{\mathbf{k}\lambda}(\mathbf{p}\cdot\mathbf{e}_{\mathbf{k}\lambda})|n,\text{vac}\rangle,\end{aligned} \tag{3.21}$$

we are led to interpret ΔE_n in terms of an emission process $n \rightarrow m+\gamma$ followed by the absorption process $m+\gamma \rightarrow n$, where γ denotes a photon. This emission and absorption of "virtual photons" is indicated diagrammatically in Figure 3.2.

We have ignored the contribution of the interaction $(e^2/2mc^2)\mathbf{A}^2$ to ΔE_n. Since this term does not involve atomic operators, it contributes the same energy to every state $|n,\text{vac}\rangle$,

$$\Delta E_o = \langle n,\text{vac}|\frac{e^2}{2mc^2}\mathbf{A}^2|n,\text{vac}\rangle$$

$$= \frac{e^2}{2mc^2}\sum_{\mathbf{k}\lambda}\sum_{\mathbf{k}'\lambda'}\left(\frac{2\pi\hbar c^2}{\omega_k V}\right)^{1/2}\left(\frac{2\pi\hbar c^2}{\omega'_k V}\right)^{1/2}$$
$$\times \langle \mathrm{vac}|a_{\mathbf{k}'\lambda'}a^{\dagger}_{\mathbf{k}\lambda}|\mathrm{vac}\rangle \mathbf{e}_{\mathbf{k}'\lambda'}\cdot\mathbf{e}_{\mathbf{k}\lambda}$$
$$= \frac{e^2}{2mc^2}\sum_{\mathbf{k}\lambda}\left(\frac{2\pi\hbar c^2}{\omega_k V}\right) = \frac{e^2\hbar}{\pi mc^3}\int_0^\infty d\omega\omega, \qquad (3.22)$$

and therefore it does not affect observable *frequency* shifts and may be ignored. This term is indicated diagrammatically in Figure 3.3.

Both ΔE_n and ΔE_o are seen to be infinite. This is especially problematic for ΔE_n, since this presumably corresponds to the Lamb shift, which experiment shows to be not infinite but *small*. It was the resolution of this dilemma that "fundamentally changed the character of theoretical physics" (Dirac, 1989). The first person to calculate a finite value for the Lamb shift was Bethe (1947), and we now turn our attention to his calculation.

3.5 Bethe's Mass Renormalization

The energy of a *free* electron due to its coupling to the field may be obtained from (3.20) by taking the limit in which all the transition frequencies $\omega_{nm} \to 0$. Thus

$$\Delta E_n^{\mathrm{free}} = -\frac{2\alpha}{3\pi}\left(\frac{1}{mc}\right)^2\sum_m |\mathbf{p}_{mn}|^2\int_0^\infty dE \qquad (3.23)$$

is the expectation value in state n of the operator corresponding to the energy of a free electron due to its coupling to the field. (As noted earlier,

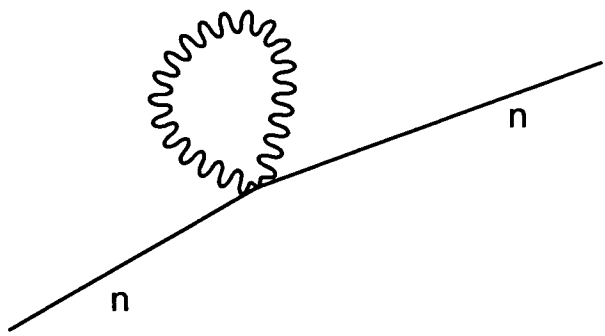

Figure 3.3: Diagrammatic representation of (3.22).

the contribution ΔE_0 is the same for every state, free or bound, and may be neglected for our purposes.)

Now it is reasonable that the observed level shift for an atom in state n should be $\Delta E_n - \Delta E_n^{\text{free}}$, the difference between the shift in the electron energy when it is bound and when it is free:

$$\begin{aligned}\Delta E_n^{\text{obs}} &= \Delta E_n - \Delta E_n^{\text{free}} \\ &= \frac{2\alpha}{3\pi}\left(\frac{1}{mc}\right)^2 \sum_m |\mathbf{p}_{mn}|^2 \left[\int_0^\infty \frac{dE E}{E_n - E_m - E} + \int_0^\infty dE\right] \\ &= \frac{2\alpha}{3\pi}\left(\frac{1}{mc}\right)^2 \sum_m |\mathbf{p}_{mn}|^2 (E_n - E_m) \int_0^\infty \frac{dE}{E_n - E_m - E} \,.\end{aligned} \tag{3.24}$$

This expression is still infinite but, unlike (3.20), the divergence is "only" logarithmic. The subtraction of ΔE_n^{free} from ΔE_n has thus reduced the divergence from linear to logarithmic. This subtraction was done by Bethe (1947). He correctly suggested that in a relativistic theory, where ΔE_n and ΔE_n^{free} themselves turn out to diverge only logarithmically, the subtraction of ΔE_n^{free} would produce a finite value for ΔE_n^{obs}.

Bethe assumed that the main part of the Lamb shift was due to the interaction of the electron with vacuum field modes of frequency small enough to justify a nonrelativistic approach. In this case it is reasonable to cut off the upper limit of integration in (3.24) by some $E_{\max}$, which Bethe took to be mc^2. Then

$$\begin{aligned}\Delta E_n^{\text{obs}} &\to \frac{2\alpha}{3\pi}\left(\frac{1}{mc}\right)^2 \sum_m |\mathbf{p}_{mn}|^2 (E_n - E_m) \int_0^{mc^2} \frac{dE}{E_n - E_m - E} \\ &\cong \frac{2\alpha}{3\pi}\left(\frac{1}{mc}\right)^2 \sum_m |\mathbf{p}_{mn}|^2 (E_m - E_n) \log \frac{mc^2}{|E_m - E_n|}\end{aligned} \tag{3.25}$$

for $mc^2 >> |E_n - E_m|$. Since the argument of the logarithm is accordingly very large, Bethe replaced the logarithm by an average value, independent of m, as a first approximation:

$$\Delta E_n^{\text{obs}} \cong \frac{2\alpha}{3\pi}\left(\frac{1}{mc}\right)^2 \log \frac{mc^2}{|E_m - E_n|_{\text{avg}}} \sum_m |\mathbf{p}_{mn}|^2 (E_m - E_n). \tag{3.26}$$

Now

$$\sum_m |\mathbf{p}_{mn}|^2 (E_m - E_n) = \sum_m \langle n|\mathbf{p}|m\rangle \cdot \langle m|\mathbf{p}|n\rangle (E_m - E_n)$$

$$
\begin{aligned}
&= \sum_m \langle n|[\mathbf{p}, H_A]|m\rangle \cdot \langle m|\mathbf{p}|n\rangle \\
&= -i\hbar\langle n|\nabla V \cdot \mathbf{p}|n\rangle = -\frac{1}{2}i\hbar\langle n|[\nabla V, \mathbf{p}]|n\rangle \\
&= \frac{1}{2}\hbar^2\langle n|\nabla^2 V|n\rangle = \frac{1}{2}\hbar^2 \int d^3r |\psi_n(\mathbf{r})|^2 \nabla^2 V(\mathbf{r}),
\end{aligned}
\tag{3.27}
$$

where V is the binding potential. ($H_A = p^2/2m + V$) For the Coulomb potential $V = -Ze^2/r$, we have $\nabla^2 V = 4\pi Ze^2\delta^3(\mathbf{r})$ and

$$
\sum_m |\mathbf{p}_{mn}|^2 (E_m - E_n) = 2\pi\hbar^2 e^2 Z |\psi_n(0)|^2, \tag{3.28}
$$

$$
\Delta E_n^{\text{obs}} \cong \frac{4\alpha Z}{3}\left(\frac{e\hbar}{mc}\right)^2 |\psi_n(0)|^2 \log\frac{mc^2}{|E_m - E_n|_{\text{avg}}} \,. \tag{3.29}
$$

This expression already exhibits an important element of truth: the Lamb shift should be largest for s states, for which $|\psi_n(0)|^2 \neq 0$. For an s state with principal quantum number n, $|\psi_n(0)|^2 = (Z/na_o)^3/\pi$ and

$$
\Delta E_{ns}^{\text{obs}} \cong \frac{8\alpha^3 Z^4}{3\pi n^3} R_\infty \log\frac{mc^2}{|E_m - E_n|_{\text{avg}}} \,, \tag{3.30}
$$

where R_∞ is the Rydberg unit of energy ($e^2/2a_o \cong 13.6$ eV) for infinite nuclear mass. Using a numerical estimate of 17.8 R_∞ for the average excitation energy $|E_n - E_m|_{\text{avg}}$ defined by[2]

$$
\log|E_m - E_n|_{\text{avg}} = \frac{\sum_m |\mathbf{p}_{mn}|^2 (E_m - E_n)\log|E_m - E_n|}{\sum_m |\mathbf{p}_{mn}|^2 (E_m - E_n)} \,, \tag{3.31}
$$

Bethe obtained for the $2s$ state of hydrogen a level shift in excellent agreement with experiment:

$$
\Delta E_{2s}^{\text{obs}} \cong 1040 \text{ MHz}. \tag{3.32}
$$

[2]The sums in this expression include continuum states. In fact the continuum states make a larger contribution to the average excitation energy than discrete states. This explains why the average excitation energy turns out to be so large compared with bound–bound transition frequencies. See H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Springer-Verlag, Berlin, 1957), pp. 318–320.

The crux of Bethe's calculation is the "mass renormalization" implied by the subtraction of ΔE_n^{free} from ΔE_n in (3.24). Recall from Appendix D that a charged particle has an "electromagnetic mass" δm due to its own radiation reaction field. The nonrelativistic calculation in Appendix D gives

$$\delta m = \frac{4\alpha}{3\pi c^2}\int_0^\infty dE. \tag{3.33}$$

As discussed in Chapter 5, the observed electron mass m is $m_o + \delta m$, where m_o is the "bare mass," i.e., the contribution to the electron mass that is not associated with radiation reaction. Although (3.33) is infinite, and δm is also infinite when calculated relativistically, we might suppose that in some future, more refined theory, δm will be finite. If we assume furthermore that $\delta m/m$ in reality is small, then the kinetic energy of the electron is

$$\frac{\mathbf{p}^2}{2m} = \frac{\mathbf{p}^2}{2(m_o+\delta m)} \cong \frac{\mathbf{p}^2}{2m_o} - \frac{\delta m}{2m^2}\mathbf{p}^2. \tag{3.34}$$

Now the basic idea behind mass renormalization is this: when we write $\mathbf{p}^2/2m$ for the electron kinetic energy in the Schrödinger equation, m is the *observed mass* ($\cong 9.1 \times 10^{-28}$ g), which includes δm. But when we "turn on" the coupling of the electron to the field in our calculations, the radiation reaction on the electron adds δm to its mass. Since we have implicitly already accounted for δm in writing the electron mass as m in the Schrödinger (or Dirac) equation with no coupling to the field, we must be careful to avoid "double counting" δm. In particular, we should *subtract* the "additional" contribution, $-(\delta m/2m^2)\mathbf{p}^2$, that we incur after coupling the electron to the field. That is, we must subtract the *self-energy*

$$\begin{aligned} -\frac{\delta m}{2m^2}\langle n|\mathbf{p}^2|n\rangle &= -\frac{4\alpha}{3\pi c^2}\left(\frac{1}{2m^2}\right)\int_0^\infty dE\langle n|\mathbf{p}^2|n\rangle \\ &= -\frac{2\alpha}{3\pi}\left(\frac{1}{mc}\right)^2\sum_m |\mathbf{p}_{mn}|^2\int_0^\infty dE \\ &= \Delta E_n^{\text{free}} \end{aligned} \tag{3.35}$$

from the calculated shift in E_n arising from the coupling of the electron to the radiation field. This is exactly what Bethe did in order to reduce the order of divergence of ΔE_n.

The idea behind renormalization is attributable to Kramers (Dresden, 1987) and also to Weisskopf (1936). Bethe applied it to the Lamb shift immediately after a conference on Shelter Island, where the Lamb–Retherford experiments and the theoretical difficulties with infinities in electrodynamics were discussed (Bethe, 1989):

> ... I thought that it ought to be possible to get Lamb's result by applying the idea of Kramers. So on the train from Shelter Island ... I wrote down some ... equations ... and found out that the effect on the $2s$-state or any state of hydrogen would involve the logarithm of the energy ... Stupidly or boldly, I just assumed that the higher energy was mc^2, and with this assumption, I got about the right answer. Of course, I was afraid that I might have made a mistake by a factor of 2 ... after all one cannot remember factors of 2 on a train. So the next morning, as early as I could, I looked for Heitler's book in the General Electric library, and found that I had not made a mistake. Indeed I got a result of about a thousand megacycles which was about the right answer.

In his Nobel lecture (1966) Feynman called Bethe's estimate "the most important discovery in the history of quantum electrodynamics."

Renormalizability, like Lorentz invariance and gauge invariance, is presently believed to be required of any fundamental theory of physics. However, dissatisfaction with renormalization has been expressed at various times by many physicists, including Dirac (1978), who felt that "This is just not sensible mathematics. Sensible mathematics involves neglecting a quantity when it turns out to be small — not neglecting it just because it is infinitely great and you do not want it!"

On the other hand, it can be argued that mass renormalization, for instance, would be necessary to avoid double counting in calculations *even if the electromagnetic mass δm turned out to be finite.* It can also be argued that δm in a more refined theory would turn out to be small, and that mass differences between the particles π^+ and π^0, or K^+ and K^0, etc., are "almost certainly electromagnetic in origin" (Feynman, 1961).

Discussion of these matters further would take us too far from our present subject. We shall return to the numerical value of the Lamb shift later in connection with vacuum polarization, but in the next few sections we wish to develop a more physical understanding of the dominant (nonrelativistic) contribution to the Lamb shift calculated by Bethe.

3.6 Welton's Interpretation

Welton (1948) interpreted the Lamb shift as follows. The vacuum field causes the position of the electron to fluctuate. The fluctuation $\Delta\mathbf{r}$ is determined by $m\Delta\ddot{\mathbf{r}} = e\mathbf{E}_0$, where $\mathbf{E}_0$ is the zero-point electric field. If we make a Fourier decomposition of both $\mathbf{E}_0$ and $\Delta\mathbf{r}$, then

$$\Delta\mathbf{r}_\omega = -\frac{e}{m\omega^2}\mathbf{E}_{0,\omega} \tag{3.36}$$

gives the component of $\Delta\mathbf{r}$ at the frequency ω. Thus

$$\langle(\Delta\mathbf{r}_\omega)^2\rangle = \frac{e^2}{m^2\omega^4}\langle\mathbf{E}^2_{o,\omega}\rangle, \tag{3.37}$$

where the expectation values are over the vacuum state of the field. Since $(\omega^2/\pi^2c^3)d\omega$ is the number of field modes per unit volume in the frequency interval $[\omega, \omega + d\omega]$, and each mode has a zero-point energy $\frac{1}{2}\hbar\omega$, we have

$$\frac{1}{2}\hbar\omega\left(\frac{\omega^2}{\pi^2c^3}\right)d\omega = \frac{1}{8\pi}[\langle\mathbf{E}^2_{o,\omega}\rangle + \langle\mathbf{B}^2_{o,\omega}\rangle]d\omega = \frac{1}{4\pi}\langle\mathbf{E}^2_{o,\omega}\rangle d\omega \tag{3.38}$$

and

$$\begin{aligned}\langle(\Delta\mathbf{r})^2\rangle &= \int_0^\infty d\omega\left(\frac{e^2}{m^2\omega^4}\right)\langle\mathbf{E}^2_{o,\omega}\rangle = \frac{2\alpha}{\pi}\left(\frac{\hbar}{mc}\right)^2\int_0^\infty\frac{d\omega}{\omega} \\ &= \frac{2\alpha}{\pi}\left(\frac{\hbar}{mc}\right)^2\int_0^\infty\frac{dE}{E}\,,\end{aligned} \tag{3.39}$$

where again α is the fine structure constant and $\hbar/mc$ is the electron Compton radius divided by 2π.

Now the fluctuation in $\mathbf{r}$ causes the potential energy $V(\mathbf{r})$ to fluctuate,

$$V(\mathbf{r}+\Delta\mathbf{r}) = V(\mathbf{r}) + \Delta\mathbf{r}\cdot\nabla V(\mathbf{r}) + \frac{1}{6}(\Delta\mathbf{r})^2\nabla^2V(\mathbf{r}) + \ldots \tag{3.40}$$

for a spherically symmetric potential, so that an electron in state n should experience an energy shift with leading term

$$\begin{aligned}\Delta E'_n &= \frac{1}{6}\langle(\Delta\mathbf{r})^2\rangle\langle n|\nabla^2V(\mathbf{r})|n\rangle = \frac{1}{6}\langle(\Delta\mathbf{r})^2\rangle 4\pi Ze^2|\psi_n(0)|^2 \\ &= \frac{8\alpha^3Z^4}{3\pi n^3}R_\infty\int_0^\infty\frac{dE}{E}\end{aligned} \tag{3.41}$$

for an s state with principal quantum number n. This is infinite, but if we replace the upper limit of integration by mc^2 in this nonrelativistic model, and the lower limit by Bethe's average excitation energy, then we recover exactly Bethe's expression (3.30) for the Lamb shift.

Note that $E'_n = 0$ for a free electron ($\nabla^2V = 0$), so that there is no need here to subtract away a free-electron contribution in order to obtain an observable shift. That is, no mass renormalization is necessary in Welton's heuristic approach to the Lamb shift.

The steps leading to (3.41) are the essence of Welton's interpretation of the Lamb shift, and this interpretation is mentioned in many textbooks in advanced quantum mechanics. Welton's argument seems to leave little doubt about the "reality" of the vacuum fluctuations of the electromagnetic field.

3.7 A Feynman Interpretation of the Lamb Shift

Thus far we have described two ways of thinking about the role of the vacuum field in the Lamb shift: we can explain the level shift as the result of emission into and re-absorption from the vacuum of virtual photons (Figure 3.2), or as the result of the fluctuations in the position of an electron due to the fluctuations in the vacuum electric field. In this section and the next we descibe two variations on this theme.

The first is a simplification of an argument due originally to Feynman (1961) (Power 1966). Consider a dilute gas of N atoms per unit volume in a large box of volume V. Since the allowed wavelengths are fixed by the dimensions of the box, the effect of the refractive index of the gas is to change the frequencies ω_k to $\omega_k/n(\omega_k)$, where $n(\omega_k)$ is the index at ω_k. The change in the zero-point field energy due to the presence of the atoms is therefore

$$\Delta E = \sum_{\mathbf{k}\lambda} \frac{1}{2}\frac{\hbar\omega_k}{n(\omega_k)} - \sum_{\mathbf{k}\lambda} \frac{1}{2}\hbar\omega_k \cong -\sum_{\mathbf{k}\lambda}[n(\omega_k)-1]\frac{1}{2}\hbar\omega_k \tag{3.42}$$

for $n(\omega_k) \cong 1$. Now $n(\omega_k)$ is given, for a dilute gas of atoms in level n, by[3]

$$n(\omega_k) \cong 1 + \frac{4\pi N}{3\hbar}\sum_m \frac{\omega_{mn}|\mathbf{d}_{mn}|^2}{\omega_{mn}^2 - \omega_k^2}\ , \tag{3.43}$$

where $\mathbf{d}_{mn}$ is the $m \leftrightarrow n$ transition dipole moment. Thus

$$\begin{aligned}\Delta E_n &\cong -\frac{2\pi N}{3}\sum_{\mathbf{k}\lambda}\omega_k \sum_m \frac{\omega_{mn}|\mathbf{d}_{mn}|^2}{\omega_{mn}^2 - \omega_k^2} \\ &\to -\frac{2NV}{3\pi c^3}\sum_m \omega_{mn}|\mathbf{d}_{mn}|^2 \int_0^\infty \frac{d\omega\omega^3}{\omega_{mn}^2 - \omega^2}\ .\end{aligned} \tag{3.44}$$

To obtain an observable shift in level n we subtract from this expression the change in zero-point energy due to N *free* electrons per unit volume in the box. This is obtained by ignoring ω_{mn}^2 compared with ω_k^2 in (3.44), i.e., by taking a limit of effectively continuous electron energies:

$$\Delta E_o = \frac{2NV}{3\pi c^3}\sum_m \omega_{mn}|\mathbf{d}_{mn}|^2 \int_0^\infty d\omega\omega\ . \tag{3.45}$$

[3] See, for instance, P. W. Milonni and J. H. Eberly, *Lasers* (Wiley, New York, 1988), Chapter 7.

The Thomas–Reiche–Kuhn sum rule,[4] $\sum_m \omega_{mn}|\mathbf{d}_{mn}|^2 = 3\hbar e^2/2m$, allows us to write this free-electron energy as

$$\Delta E_o = (NV)\frac{e^2\hbar}{\pi mc^3}\int_0^\infty d\omega\omega \ . \tag{3.46}$$

which is just the vacuum expectation value of the energy $(e^2/2mc^2)\mathbf{A}^2$ for NV electrons. [See equation (3.22).] The observable shift in level n should therefore be

$$\begin{aligned}\Delta E_n^{\text{obs}} &= -\frac{2NV}{3\pi c^3}\sum_m \omega_{mn}|\mathbf{d}_{mn}|^2\int_0^\infty d\omega[\frac{\omega^3}{\omega_{mn}^2-\omega^2}+\omega] \\ &= -\frac{2NV}{3\pi c^3}\sum_m \omega_{mn}^3|\mathbf{d}_{mn}|^2\int_0^\infty \frac{d\omega\omega}{\omega_{mn}^2-\omega^2} \\ &\to \frac{2NV}{3\pi c^3}\sum_m \omega_{mn}^3|\mathbf{d}_{mn}|^2\log\frac{mc^2}{|E_m-E_n|}\end{aligned} \tag{3.47}$$

when we introduce a high-frequency cutoff $mc^2/\hbar$.

Finally we recall that[5] $|\mathbf{p}_{mn}|^2 = m^2\omega_{mn}^2|\mathbf{x}_{mn}|^2 = (m^2\omega_{mn}^2/e^2)|\mathbf{d}_{mn}|^2$ and write (3.47) in the form

$$\Delta E_n^{\text{obs}} = (NV)\frac{2\alpha}{3\pi}\left(\frac{1}{mc}\right)^2\sum_m (E_m-E_n)|\mathbf{p}_{mn}|^2\log\frac{mc^2}{|E_m-E_n|} \ , \tag{3.48}$$

which is exactly Bethe's expresssion (3.25) obtained after mass renormalization when we take $NV = 1$, i.e., when we let our original box contain one atom. Note that, as in Welton's argument and for basically the same reason, no mass renormalization is required in this approach.

3.8 The Lamb Shift as a Stark Shift

There is yet another interpretation of the "Bethe log." Consider the energy $W = -\frac{1}{2}\mathbf{d}\cdot\mathbf{E}$ associated with a dipole moment $\mathbf{d}$ induced by an electric field $\mathbf{E}$.[6] Writing $\mathbf{d}_\omega = \alpha(\omega)\mathbf{E}_\omega$ for the Fourier component of the dipole moment induced by the Fourier component $\mathbf{E}_\omega$ of the field, where $\alpha(\omega)$ is

[4] See, for instance, J. J. Sakurai, *Advanced Quantum Mechanics* (Addison–Wesley, Reading, Mass., 1976), p. 74.

[5] Ibid., p. 42.

[6] The factor 1/2 is due to the fact that the dipole moment is induced rather than permanent. See, for instance, J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975), p. 161.

the polarizability, we have $W = -\frac{1}{2}\alpha(\omega)E_\omega^2$ and, if there is a continuous distribution of field frequencies,

$$W = -\frac{1}{2}\int \alpha(\omega)[4\pi\rho(\omega)d\omega], \tag{3.49}$$

where $\rho(\omega)$ is the spectral energy density of the field, $\mathbf{E}_\omega^2 = 4\pi\rho(\omega)d\omega$. For an atom in level n, therefore, we expect a level shift

$$\Delta E_n = -2\pi \int_0^\infty d\omega \alpha_n(\omega)\rho(\omega) \tag{3.50}$$

due to an applied field of spectral energy density $\rho(\omega)$, where $\alpha_n(\omega)$ is the polarizability for level n and is given by the Kramers–Heisenberg formula,[7]

$$\alpha_n(\omega) = \frac{2}{3\hbar}\sum_m \frac{\omega_{mn}|\mathbf{d}_{mn}|^2}{\omega_{mn}^2 - \omega^2}\,. \tag{3.51}$$

For a monochromatic field, equation (3.50) reduces to the standard formula for the second-order Stark shift produced by an external field.

For an atom in the vacuum we use the spectral energy density $\rho_o(\omega) = \hbar\omega^3/2\pi^2c^3$ of the zero-point field and obtain the level shift

$$\Delta E_n = -\frac{2}{3\pi c^3}\sum_m \omega_{mn}|\mathbf{d}_{mn}|^2 \int_0^\infty \frac{d\omega\omega^3}{\omega_{mn}^2 - \omega^2}\,. \tag{3.52}$$

This is identical to equation (3.44) in the case of one atom ($NV = 1$). Therefore we can regard the Lamb shift as a Stark shift produced by the vacuum electromagnetic field.

The equivalence of this interpretation of the Lamb shift to that given in the preceding section follows from the relation

$$n(\omega) = 1 + 2\pi N\alpha(\omega) \tag{3.53}$$

between the refractive index and the polarizability for a gas with $n(\omega) \cong 1$. These interpretations of the Lamb shift can be "dressed up" by relating $\alpha(\omega)$ to the real part of the forward scattering amplitude $f(\omega)$ $[\propto \omega^2\alpha(\omega)]$ for a photon of frequency ω, but this is hardly necessary to bring out the point that the Bethe log may be attributed to the coupling of the atom to the vacuum radiation field.

[7] See, for instance, A. S. Davydov, *Quantum Mechanics* (Pergamon Press, Oxford, 1965), pp. 316–321.

3.9 Retardation

The Bethe log arising from vacuum field fluctuations accounts for all but a few percent of the $2s_{1/2} - 2p_{1/2}$ Lamb shift in hydrogen. The Lamb shift provides one of the most delicate tests of QED, and various other effects contributing to it, such as vacuum polarization, finite nuclear mass, etc., must be accounted for in any detailed comparison with experiment. However, for a basic understanding of why there is a Lamb shift and for an estimate of its magnitude in hydrogen, it is sufficient to concentrate on the Bethe log. For this reason we have discussed various physical interpretations of this contribution, all of them involving the vacuum electromagnetic field.

It is obviously of interest to determine the effect of higher order corrections to the Bethe log. One correction is to go beyond the dipole approximation in which the factors $e^{\pm i\mathbf{k}\cdot\mathbf{x}}$ in the field are dropped. This results in the replacement of (3.19) by

$$\Delta E_n = \frac{2\pi e^2}{m^2}\frac{1}{V}\sum_{\mathbf{k}\lambda}\frac{1}{\omega_k}\sum_m \frac{\langle n|\mathbf{p}\cdot\mathbf{e}_{\mathbf{k}\lambda}e^{i\mathbf{k}\cdot\mathbf{x}}|m\rangle\langle m|e^{-i\mathbf{k}\cdot\mathbf{x}}\mathbf{p}\cdot\mathbf{e}_{\mathbf{k}\lambda}|n\rangle}{\omega_{nm}-\omega_k}\,, \tag{3.54}$$

which, unlike (3.19), is logarithmically divergent *without mass renormalization* (Au and Feinberg, 1974). We can see this by writing

$$\sum_m \frac{\mathbf{p}\cdot\mathbf{e}_{\mathbf{k}\lambda}e^{i\mathbf{k}\cdot\mathbf{x}}|m\rangle\langle m|e^{-i\mathbf{k}\cdot\mathbf{x}}\mathbf{p}\cdot\mathbf{e}_{\mathbf{k}\lambda}}{\omega_{nm}-\omega_k} = \hbar\sum_m \mathbf{p}\cdot\mathbf{e}_{\mathbf{k}\lambda}e^{i\mathbf{k}\cdot\mathbf{x}}G_n|m\rangle\langle m| \times e^{-i\mathbf{k}\cdot\mathbf{x}}\mathbf{p}\cdot\mathbf{e}_{\mathbf{k}\lambda} \tag{3.55}$$

in (3.54), where the operator $G_n \equiv (E_n - \mathbf{p}^2/2m - V - \hbar\omega_k)^{-1}$, with $(\mathbf{p}^2/2m + V)|m\rangle = E_m|m\rangle$. Then the general identity

$$e^{i\mathbf{k}\cdot\mathbf{x}}F(\mathbf{p})e^{-i\mathbf{k}\cdot\mathbf{x}} = F(\mathbf{p}-\hbar\mathbf{k}) \tag{3.56}$$

gives

$$e^{i\mathbf{k}\cdot\mathbf{x}}G_n e^{-i\mathbf{k}\cdot x} = [E_n - \frac{1}{2m}(\mathbf{p}-\hbar\mathbf{k})^2 - V - \hbar\omega_k]^{-1}\,. \tag{3.57}$$

The effect of the retardation, i.e., of not making the dipole approximation, is then to replace ω by ω^2 for large frequencies in the denominator of the integrand in (3.20). This leads to logarithmic rather than linear divergence.

Physically, the effect of retardation is to give *recoil* of the electron in photon emission and absorption. The replacement of $\mathbf{p}$ by $\mathbf{p} - \hbar\mathbf{k}$ in the

intermediate electron momenta then reflects the conservation of linear momentum in the interaction between electrons and photons.

We can aproach the free-particle self-energy with retardation included by taking $E_n = \mathbf{p}^2/2m$ and $V = 0$ in (3.57), and ignoring the term $\mathbf{k} \cdot \mathbf{p}$ in order to obtain from (3.56) a contribution proportional to $\mathbf{p}^2$ for small $\mathbf{p}$. Then (3.54) gives

$$\Delta E_n^{\text{free}} = -\frac{2\pi e^2}{m^2}\frac{1}{V}\sum_{\mathbf{k}\lambda}\frac{1}{\omega_k}\frac{\langle n|(\mathbf{p}\cdot\mathbf{e}_{\mathbf{k}\lambda})^2|n\rangle}{\omega_k + \hbar\omega_k^2/2mc^2} = -\frac{\delta m}{2m^2}\langle n|\mathbf{p}^2|n\rangle, \qquad (3.58)$$

where

$$\delta m = \frac{4e^2}{3\pi c^3}\int_0^\infty \frac{d\omega\omega}{\omega + \hbar\omega^2/2mc^2}\ , \qquad (3.59)$$

which is logarithmically divergent as opposed to the linear divergence of the result (3.33) obtained without retardation.

Using a momentum representation of G_n (Schwinger, 1964; see also Lieber, 1968), Au and Feinberg (1974) have numerically computed the mass-renormalized shifts $\Delta E_n - (\delta m/2m^2)\langle n|\mathbf{p}^2|n\rangle$, with ΔE_n and δm given by (3.54) and (3.59), respectively, for the $2s$ and $2p$ levels of hydrogen with retardation. Each term diverges logarithmically, and the difference is finite without any high-frequency cutoff. They obtained $\Delta E_{2s} = 931.1$ MHz and $\Delta E_{2s} - \Delta E_{2p} = 950.3$ MHz. (When the $\mathbf{k} \cdot \mathbf{p}$ term was kept in the denominator of (3.58) in the evaluation of ΔE_n^{free}, they obtained $\Delta E_{2s} = 1330$ MHz and $\Delta E_{2s} - \Delta E_{2p} = 996.6$ MHz.) The nonrelativistic computation of the Lamb shift with retardation included, therefore, gives a finite value, but this value is significantly different from the experimentally observed Lamb shift (see also Grotch, 1981).

The nonrelativistic theory with retardation as just described, of course, involves photon frequencies at which nonrelativistic theory breaks down. In this sense the theory is inconsistent. However, it does show that retardation is important in the numerical value of the Lamb shift, and furthermore it provides some insight into why the nonrelativistic, nonretarded Bethe log with a high-frequency cutoff works so well (Au and Feinberg, 1974): the dominant contribution to the Bethe log comes from frequencies that are too small for retardation to be important, while, for high frequencies, where Bethe introduced a cutoff, there is an *effective* cutoff resulting from retardation.

It is worth noting that the $\Delta E_{2s} - \Delta E_{2p}$ separation is convergent in the nonrelativistic theory with retardation even without mass renormalization.

3.10 Another Look at the Casimir Force

In Section 2.7 we obtained the Casimir force between two conducting plates in the conventional way, by calculating the difference between the zero-point field energies for finite and infinite plate separations. Having interpreted the Lamb shift in different ways based on vacuum field fluctuations, we now turn to an alternative interpretation of the Casimir force.

The idea here is that the virtual photons of the vacuum carry linear momentum $\frac{1}{2}\hbar\mathbf{k}$; recall equation (2.60). Then the reflections off the plates of the zero-point field outside the plates act to push the plates together, while reflections of the field confined between the plates push them apart. Loosely speaking, there are more field modes outside the plates than inside, since only certain discrete frequencies are allowed between the plates. The net effect of the zero-point radiation pressure is then to push the plates together. We shall now show that the force calculated in this way is exactly the Casimir force (Milonni, Cook, and Goggin, 1988).

Consider the radiation pressure exerted by a plane wave incident normally on a plate. This pressure is twice the energy u per unit volume of the incident field (Section 1.2). If the wave has an angle of incidence θ, however, the radiation pressure is reduced to $P = F/A = 2u\cos^2\theta$. There are two factors of $\cos\theta$ here because (1) the normal component of the linear momentum imparted to the plate is proportional to $\cos\theta$, and (2) the element of area A is increased by $(\cos\theta)^{-1}$ compared with the case of normal incidence.

Between the plates the modes formed by reflections off the plates obviously act to push the plates apart. A mode of frequency ω contributes a pressure

$$P = 2(\frac{1}{2})(\frac{1}{2}\hbar\omega)V^{-1}\cos^2\theta = \frac{\hbar\omega}{2V}\frac{k_z^2}{k^2}\ , \tag{3.60}$$

where, as usual, $k = \omega/c$ and V is a quantization volume. A factor 1/2 has been inserted because the zero-point energy of each mode is divided equally between waves propagating toward or away from each plate. For large plates, k_x and k_y take on a continuum of values, whereas $k_z = n\pi/d$, where n is a positive integer. Adding the contributions from all modes of the space between the plates, we have the total outward pressure

$$P_{\text{out}} = \frac{\hbar c}{\pi^2 d}\sum_{n=1}^{\infty}\int_0^\infty dk_x\int_0^\infty dk_y\frac{(n\pi/d)^2}{[k_x^2+k_y^2+(n\pi/d)^2]^{1/2}} \tag{3.61}$$

on each plate. In writing this expression a factor of 2 has been included to allow for the two independent polarizations. The replacement of sums by

integrals over k_x and k_y brings in a factor $(L/\pi)^2 = V/\pi^2 d$, as in Section 2.7.

The field outside the "resonator" formed by the plates has a continuum of allowed frequencies. These modes obviously act to push the plates together by reflections off the plates. The total inward pressure exerted by these modes may be obtained from (3.61) by replacing $\sum_n$ by $(d/\pi)\int dk_z$:

$$P_{\text{in}} = \frac{\hbar c}{\pi^3}\int_0^\infty dk_x \int_0^\infty dk_y \int_0^\infty dk_z \frac{k_z^2}{(k_x^2+k_y^2+k_z^2)^{1/2}} . \tag{3.62}$$

Both P_{out} and P_{in} are infinite, but it is only their difference that is physically meaningful. After some simple algebra we can write this difference as

$$P_{\text{out}} - P_{\text{in}} = \frac{\pi^2\hbar c}{4d^4}\left[\sum_{n=1}^\infty n^2 \int_0^\infty \frac{dx}{(x+n^2)^{1/2}} - \int_0^\infty du u^2 \int_0^\infty \frac{dx}{(x+u^2)^{1/2}}\right]. \tag{3.63}$$

Application of the Euler–Maclaurin summation formula as in Section 2.7 then leads to the Casimir result

$$P_{\text{out}} - P_{\text{in}} = -\frac{\pi^2\hbar c}{240 d^4} \tag{3.64}$$

for the force per unit area between the plates.

We can therefore regard the Casimir force as a consequence of the radiation pressure associated with the zero-point energy $\frac{1}{2}\hbar\omega$ per mode of the field. This interpretation is directly connected to the conventional one (Section 2.7) through the Maxwell stress tensor for the quantized field (Milonni et al., 1988).[8]

3.11 Van der Waals Forces

In order to account for observed deviations from the ideal gas law, in 1873 J. D. van der Waals proposed the equation of state

$$(P + \frac{a}{V^2})(V - b) = RT \tag{3.65}$$

for 1 mole of a gas at temperature T. P and V are as usual the pressure and volume, R is the universal gas constant, and a and b are now called the *van der Waals constants,* obtained by fitting (3.65) to experimental data.

[8] See also A. E. González, *Physica* **131A**, 228 (1985).

Van der Waals interpreted the constant b as the volume excluded by two atoms. If the atoms were imagined to be spheres of radius r_o, then $b = 16\pi r_o^3/3$. The constant a was associated with an attractive force between two atoms. Van der Waals later suggested an interaction potential of the form $V(r) = -Ar^{-1}e^{-Br}$, where A and B are constants.

Much later Keesom obtained the potential $V(r) = -p_1^2 p_2^2/3kTr^6$ for two polar molecules, i.e., molecules with permanent dipole moments. Here p_1 and p_2 are the dipole moments of the two molecules and $V(r)$ is obtained as a consequence of molecular rotations. There is an attractive force because attractive orientations are statistically favored over repulsive ones.

Debye and others recognized that more general attractive forces must exist between molecules, since gases of nonpolar molecules have nonvanishing values of the van der Waals constant a. Moreover a temperature-independent potential was needed. Debye noted that many molecules have a permanent quadrupole moment, which can induce a dipole moment in a second molecule, and the resulting dipole–quadrupole force is temperature independent. Such an "induction force" occurs also if the first molecule has a permanent dipole moment. However, neither case is sufficiently general to account for the van der Waals equation of state.

London (1930) employed fourth-order quantum-mechanical perturbation theory to derive the interaction potential[9]

$$V(r) = -\frac{3\hbar\omega_o\alpha^2}{4r^6} \tag{3.66}$$

between two identical atoms (or molecules) with transition frequency ω_o between the ground and first excited levels, with α the static (zero-frequency) polarizability. London's result, which was considered a major accomplishment of the new quantum mechanics, showed that there is a general force of attraction between two molecules even if neither has a permanent moment; it is necessary only that a dipole moment can be *induced* in each molecule, i.e., that each molecule is polarizable ($\alpha \neq 0$). And London's result, unlike Keesom's, is temperature independent.

Since it involves the polarizability, which in turn is related to the refractive index and dispersion [cf. (3.53)], London's force is often called the *dispersion force*. Dispersion forces, together with the orientation and induction forces of Keesom and Debye, are now regarded as three general types of van der Waals forces. In this section we will consider the origin of the dispersion force between two neutral polarizable particles, and show that this type of van der Waals force may be attributed to zero-point energy.

[9]Before the work of London, S. C. Wang [*Phys. Zs.* **28**, 663 (1927)] presented somewhat indirect quantum-mechanical arguments for an r^{-6} interaction between two hydrogen atoms.

In fact London originally proposed such an interpretation. The essence of the argument is as follows. Consider two identical dipole oscillators of frequency ω_o coupled through their near fields. For this system we write the equations of motion

$$\ddot{x}_1 + \omega_o^2 x_1 = K x_2, \tag{3.67}$$

$$\ddot{x}_2 + \omega_o^2 x_2 = K x_1, \tag{3.68}$$

with $K = qe^2/mr^3$, q being the dipole–dipole orientation factor, $3(\hat{\mu}_1 \cdot \mathbf{s})(\hat{\mu}_2 \cdot \mathbf{s}) - \hat{\mu}_1 \cdot \hat{\mu}_2$, where $\hat{\mu}_1, \hat{\mu}_2$ are unit vectors in the directions of the dipole moments and $\mathbf{s}$ is a unit vector pointing from one dipole to the other. Two points about these equations are worth noting. First, since atoms that remain with high probability in their ground states are accurately represented for many purposes as harmonic oscillators, these equations provide a reasonable qualitative description of the coupling between two ground-state atoms.[10] Second, we are not assuming permanent dipole moments; equations (3.67) and (3.68) can be thought of as operator equations, with the expectation values $\langle x_1 \rangle = \langle x_2 \rangle = 0$ implying a vanishing permanent dipole moment. In fact the only thing of interest for the present discussion is that the normal mode frequencies of this coupled oscillator system are given by

$$\omega_\pm = (\omega_o^2 \pm K)^{1/2} . \tag{3.69}$$

The quantum-mechanical ground-state energy of the system is

$$E = \frac{1}{2}\hbar(\omega_+ + \omega_-) \cong \hbar\omega_o - \frac{\hbar K^2}{8\omega_o^3} \tag{3.70}$$

to lowest order in K/ω_o^2. This implies an interaction energy

$$V(r) = -\frac{\hbar}{8\omega_o^3}\left(\frac{qe^2}{mr^3}\right)^2 = -\frac{q^2\hbar\omega_o\alpha^2}{8r^6} , \tag{3.71}$$

where $\alpha = e^2/m\omega_o^2$ is the classical static polarizability. Now if we use the fact that a quantum-mechanical evaluation of q^2 gives an average value of 2, and multiply by 3 to account for the three-dimensionality of the atoms, then (3.71) yields $V(r) = -3\hbar\omega_o\alpha^2/4r^6$, which is London's result (3.66).[11]

[10] See, for instance, M. Cray, M.-L. Shih, and P. W. Milonni, *Am. J. Phys.* **50**, 1016 (1982).

[11] If the two induced dipoles are parallel to each other and perpendicular to the axis joining them, $q^2 = 1$, whereas if they are parallel and aligned along the axis, $q^2 = 4$. The orientationally averaged value of q^2 is then $(2/3)(1) + (1/3)(4) = 2$. See also P. W. Milonni and P. L. Knight, *Phys. Rev.* **A10**, 1096 (1974); **A11**, 1090 (1975).

We saw in Section 2.6 that the vacuum field is required to maintain the commutation relation, and therefore the zero-point energy, of a dipole oscillator. This suggests that the r^{-6} van der Waals interaction might be attributed physically to the fluctuating vacuum electromagnetic field. That this is so may be seen from the following argument.

As in Section 3.8 we begin with the formula for the Stark shift of an atomic energy level, but now without the assumption that the field is isotropic:

$$W_{\rm A} = -\frac{1}{2}\sum_{\mathbf{k}\lambda} \alpha_{\rm A}(\omega_k)\mathbf{E}^2_{\mathbf{k}\lambda}(\mathbf{x}_{\rm A}, t) \tag{3.72}$$

for an atom A located at $\mathbf{x}_{\rm A}$ with polarizability $\alpha_{\rm A}(\omega)$. The total field in mode $(\mathbf{k}, \lambda)$ acting on A is assumed to be the zero-point field plus the field at A produced by a second atom B:

$$\mathbf{E}_{\mathbf{k}\lambda}(\mathbf{x}_{\rm A}, t) = \mathbf{E}_{{\rm o},\mathbf{k}\lambda}(\mathbf{x}_{\rm A}, t) + \mathbf{E}_{{\rm B},\mathbf{k}\lambda}(\mathbf{x}_{\rm A}, t). \tag{3.73}$$

Then the part of $W_{\rm A}$ due to the interaction between the two atoms is

$$\begin{aligned} W_{\rm AB} &= -\frac{1}{2}\sum_{\mathbf{k}\lambda} \alpha_{\rm A}(\omega_k)[\mathbf{E}_{{\rm o},\mathbf{k}\lambda}(\mathbf{x}_{\rm A}, t)\cdot\mathbf{E}_{{\rm B},\mathbf{k}\lambda}(\mathbf{x}_{\rm A}, t) \\ &\quad + \mathbf{E}_{{\rm B},\mathbf{k}\lambda}(\mathbf{x}_{\rm A}, t)\cdot\mathbf{E}_{{\rm o},\mathbf{k}\lambda}(\mathbf{x}_{\rm A}, t)]. \end{aligned} \tag{3.74}$$

This is the only part of $W_{\rm A}$ that will involve the distance r between the atoms. We have seen in Section 3.8 that $\mathbf{E}^2_{{\rm o},\mathbf{k}\lambda}(\mathbf{x}_{\rm A}, t)$, for instance, will contribute to the Lamb shift in atom A.

Now actually the right side of (3.74) should be a vacuum expectation value involving field operators. Let us write the operator $\mathbf{E}_{{\rm o},\mathbf{k}\lambda}$ as $\mathbf{E}^{(+)}_{{\rm o},\mathbf{k}\lambda} + \mathbf{E}^{(-)}_{{\rm o},\mathbf{k}\lambda}$, where

$$\mathbf{E}^{(+)}_{{\rm o},\mathbf{k}\lambda}(\mathbf{x}_{\rm A}, t) = i\left(\frac{2\pi\hbar\omega_k}{V}\right)^{1/2} a_{\mathbf{k}\lambda}(0)e^{-i\omega_k t}e^{i\mathbf{k}\cdot\mathbf{x}_{\rm A}}\mathbf{e}_{\mathbf{k}\lambda} \tag{3.75}$$

and

$$\mathbf{E}^{(-)}_{{\rm o},\mathbf{k}\lambda}(\mathbf{x}_{\rm A}, t) = -i\left(\frac{2\pi\hbar\omega_k}{V}\right)^{1/2} a^\dagger_{\mathbf{k}\lambda}(0)e^{i\omega_k t}e^{-i\mathbf{k}\cdot\mathbf{x}_{\rm A}}\mathbf{e}_{\mathbf{k}\lambda} \tag{3.76}$$

are called the positive- and negative-frequency parts, respectively, of $\mathbf{E}_{{\rm o},\mathbf{k}\lambda}$. Since $\mathbf{E}^{(+)}_{{\rm o},\mathbf{k}\lambda}(\mathbf{x}_{\rm A}, t)|{\rm vac}\rangle = \langle{\rm vac}|\mathbf{E}^{(-)}_{{\rm o},\mathbf{k}\lambda}(\mathbf{x}_{\rm A}, t) = 0$, (3.74) is equivalent to

$$W_{\rm AB} = -\frac{1}{2}\sum_{\mathbf{k}\lambda} \alpha_{\rm A}(\omega_k)\left[\langle\mathbf{E}^{(+)}_{{\rm o},\mathbf{k}\lambda}(\mathbf{x}_{\rm A}, t)\cdot\mathbf{E}_{{\rm B},\mathbf{k}\lambda}(\mathbf{x}_{\rm A}, t)\rangle\right.$$

$$+ \langle \mathbf{E}_{\mathrm{B},\mathbf{k}\lambda}(\mathbf{x}_\mathrm{A},t) \cdot \mathbf{E}^{(-)}_{\mathrm{o},\mathbf{k}\lambda}(\mathbf{x}_\mathrm{A},t)\rangle\Big] \tag{3.77}$$

when a vacuum expectation value is taken.

The field from atom B has the same form as the classical field of an electric dipole $\mathbf{p}_\mathrm{B} = \hat{\mu}_\mathrm{B} p_\mathrm{B}(t)$, where as before $\hat{\mu}_\mathrm{B}$ is the unit vector in the direction of $\mathbf{p}_\mathrm{B}$:

$$\begin{aligned}\mathbf{E}_\mathrm{B}(\mathbf{x}_\mathrm{A},t) &= -[\hat{\mu}_\mathrm{B} - (\hat{\mu}_\mathrm{B}\cdot\mathbf{s})\mathbf{s}]\frac{1}{c^2 r}\ddot{p}_\mathrm{B}(t-\frac{r}{c}) + [3(\hat{\mu}_\mathrm{B}\cdot\mathbf{s})\mathbf{s} - \hat{\mu}_\mathrm{B}] \\ &\quad \times [\frac{1}{r^3}p_\mathrm{B}(t-\frac{r}{c}) + \frac{1}{cr^2}\dot{p}_\mathrm{B}(t-\frac{r}{c})].\end{aligned} \tag{3.78}$$

Here $\mathbf{s}$ is the unit vector pointing from atom B to atom A and now $\mathbf{E}_\mathrm{B}$ and p_B are quantum-mechanical operators.[12] The dipole moment $\mathbf{p}_\mathrm{B}$ has zero expectation value for an atom in a stationary state (the ground state in the situation of interest here), but this does not of course mean that the dipole moment of atom B is identically zero. Rather, this dipole moment *fluctuates* about zero mean due to the influence of the vacuum field at $\mathbf{x}_\mathrm{B}$:

$$\mathbf{p}_\mathrm{B}(t) = \sum_{\mathbf{k}\lambda} \alpha_\mathrm{B}(\omega_k)\left[\mathbf{E}^{(+)}_{\mathrm{o},\mathbf{k}\lambda}(\mathbf{x}_\mathrm{B},t) + \mathbf{E}^{(-)}_{\mathrm{o},\mathbf{k}\lambda}(\mathbf{x}_\mathrm{B},t)\right]. \tag{3.79}$$

Since $\mathbf{E}^{(+)}_{\mathrm{o},\mathbf{k}\lambda}|\mathrm{vac}\rangle = 0$, only the negative-frequency (creation) part of $\mathbf{E}_{\mathrm{B},\mathbf{k}\lambda}(\mathbf{x}_\mathrm{B},t)$, determined by $\mathbf{E}^{(-)}_{\mathrm{o},\mathbf{k}\lambda}(\mathbf{x}_\mathrm{B},\mathbf{t})$, will contribute to the first term in brackets in (3.77). This is easily read off from (3.78) and (3.79):

$$\begin{aligned}\mathbf{E}^{(-)}_{\mathrm{B},\mathbf{k}\lambda}(\mathbf{x}_\mathrm{A},t) &= k^3\alpha_\mathrm{B}(\omega_k)e^{-ikr}\Big\{[\mathbf{e}_{\mathbf{k}\lambda} - (\mathbf{e}_{\mathbf{k}\lambda}\cdot\mathbf{s})\mathbf{s}]\frac{1}{kr} \\ &\quad + [3(\mathbf{e}_{\mathbf{k}\lambda}\cdot\mathbf{s})\mathbf{s} - \mathbf{e}_{\mathbf{k}\lambda}] \\ &\quad [\frac{1}{(kr)^3} + \frac{i}{(kr)^2}]\Big\} E^{(-)}_{\mathrm{o},\mathbf{k}\lambda}(\mathbf{x}_\mathrm{B},t),\end{aligned} \tag{3.80}$$

where $E^{(-)}_{\mathrm{o},\mathbf{k}\lambda}(\mathbf{x}_\mathrm{B},t) \equiv \mathbf{e}_{\mathbf{k}\lambda}\cdot\mathbf{E}^{(-)}_{\mathrm{o},\mathbf{k}\lambda}(\mathbf{x}_\mathrm{B},t)$. Similarly, only $\mathbf{E}^{(+)}_{\mathrm{B},\mathbf{k}\lambda}(\mathbf{x}_\mathrm{A},t)$ will contribute to the second term in brackets in (3.77). Then, using the fact that

$$\langle \mathrm{vac}|a_{\mathbf{k}\lambda}(0)a^\dagger_{\mathbf{k}\lambda}(0)|\mathrm{vac}\rangle = 1, \tag{3.81}$$

we obtain from (3.77) the expression

$$W_\mathrm{AB} \equiv V(r) = -\frac{2\pi\hbar}{V}\mathrm{Re}\sum_{\mathbf{k}\lambda} k^3\omega_k\alpha_\mathrm{A}(\omega_k)\alpha_\mathrm{B}(\omega_k)e^{-ikr}e^{i\mathbf{k}\cdot\mathbf{r}}$$

[12] See Chapter 4 for a discussion of the correspondence between classical and quantum solutions of the Maxwell equations.

$$\times \left\{ [1 - (\mathbf{e}_{\mathbf{k}\lambda} \cdot \mathbf{s})^2] \frac{1}{kr} + [3(\mathbf{e}_{\mathbf{k}\lambda} \cdot \mathbf{s})^2 - 1][\frac{1}{(kr)^3} + \frac{i}{(kr)^2}] \right\}. \tag{3.82}$$

Note that this expression is symmetric in A and B, as it should be.

Now as usual $\sum_{\mathbf{k}\lambda} \rightarrow (V/8\pi^3) \int dk k^2 \sum_\lambda \int d\Omega_{\mathbf{k}}$, and the sum over polarizations plus the integration over solid angles about $\mathbf{k}$ are easily carried out, using the identity $\sum_\lambda (\mathbf{e}_{\mathbf{k}\lambda} \cdot \mathbf{s})^2 = 1 - (\hat{\mathbf{k}} \cdot \mathbf{s})^2$, with $\hat{\mathbf{k}} \equiv \mathbf{k}/k$:

$$\sum_\lambda \int d\Omega_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \left\{ [1 - (\mathbf{e}_{\mathbf{k}\lambda} \cdot \mathbf{s})^2] \frac{1}{kr} + [3(\mathbf{e}_{\mathbf{k}\lambda} \cdot \mathbf{s})^2 - 1][\frac{1}{(kr)^3} + \frac{i}{(kr)^2}] \right\}$$

$$= \int d\Omega_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{s}r} \left\{ [1 + (\hat{\mathbf{k}} \cdot \mathbf{s})^2] \frac{1}{kr} + [1 - 3(\hat{\mathbf{k}} \cdot \mathbf{s})^2][\frac{1}{(kr)^3} + \frac{i}{(kr)^2}] \right\}. \tag{3.83}$$

This integral is easily performed by choosing the polar (z) axis to lie along $\mathbf{s}$:

$$\frac{1}{8\pi} \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta e^{ix\cos\theta} [(1 + \cos^2\theta) \frac{1}{x} + (1 - 3\cos^2\theta)$$

$$\times (\frac{1}{x^3} + \frac{i}{x^2})] = \left(\frac{\sin x}{x^2} - \frac{i \sin x}{x^3} + \frac{\cos x}{x^3} - \frac{2 \sin x}{x^4} - \frac{3i \cos x}{x^4} \right.$$

$$\left. + \frac{3i \sin x}{x^5} - \frac{3 \cos x}{x^5} + \frac{3 \sin x}{x^6} \right), \tag{3.84}$$

with $x \equiv kr$. Then (3.82) gives

$$V(r) = -\frac{\hbar}{\pi c^6} \int_0^\infty d\omega \omega^6 \alpha_{\mathrm{A}}(\omega) \alpha_{\mathrm{B}}(\omega) G\left(\frac{\omega r}{c}\right), \tag{3.85}$$

$$G(x) \equiv \frac{\sin 2x}{x^2} + \frac{2 \cos 2x}{x^3} - \frac{5 \sin 2x}{x^4} - \frac{6 \cos 2x}{x^5} + \frac{3 \sin 2x}{x^6}. \tag{3.86}$$

For small r the dominant contribution to $V(r)$ comes from the last term in (3.86):

$$\begin{aligned} V(r) &\cong -\frac{\hbar}{\pi c^6} \frac{3c^6}{r^6} \int_0^\infty d\omega \alpha_{\mathrm{A}}(\omega) \alpha_{\mathrm{B}}(\omega) \sin \frac{2\omega r}{c} \\ &= -\frac{3\hbar}{\pi r^6} \int_0^\infty du \alpha_{\mathrm{A}}(iu) \alpha_{\mathrm{B}}(iu) e^{-2ur/c} \\ &= -\frac{3\hbar}{\pi r^6} \left(\frac{2}{3\hbar}\right)^2 \sum_m \sum_p \omega_{mn} \omega_{pn} |\mathbf{d}_{mn}|^2 |\mathbf{d}_{pn}|^2 \\ &\quad \times \int_0^\infty \frac{du e^{-2ur/c}}{(u^2 + \omega_{mn}^2)(u^2 + \omega_{pn}^2)}, \end{aligned} \tag{3.87}$$

where we have used equation (3.51) for the polarizability of an atom in state n and assumed for simplicity that the two atoms are identical. The change in the path of integration implied by the second line of (3.87), where we replace an integral along the real axis by an integral along the imaginary axis plus a (vanishing) contribution along a large quarter-circle, assumes that we do not need to concern ourselves with poles of $\alpha(\omega)$. This is in fact the case because, at a resonance frequency ω_{mn}, the real part of the polarizability is found to vanish when the $m \leftrightarrow n$ transition linewidth is accounted for.[13] If $r \to 0$, or more precisely if $r << c/|\omega_{mn}|$ for all transitions $m \leftrightarrow n$, we may replace $e^{-2ur/c}$ by 1 in (3.87), and this gives the r^{-6} form of the van der Waals potential derived by London. In fact if we assume furthermore that one particular transition $m \leftrightarrow n$ makes a dominant contribution to (3.87), then

$$V(r) \cong -\frac{3\hbar}{\pi r^6}\left(\frac{2}{3\hbar}\right)^2 \omega_o^2 |\mathbf{d}|^4 \int_0^\infty \frac{du}{(u^2+\omega_o^2)^2} = -\frac{3\hbar\omega_o\alpha^2}{4r^6}\,, \qquad (3.88)$$

where ω_o and $\mathbf{d}$ are respectively the transition frequency and dipole matrix element of this transition and $\alpha = (2/3\hbar)|\mathbf{d}|^2/\omega_o$ is the static ($\omega = 0$) polarizability of a ground-state atom in the two-state approximation in which the one transition is assumed to be dominant. This result is exactly that derived by London.

However, the r^{-6} van der Waals potential does not apply in the "retarded" regime of large interatomic separations. In a study of the stability of certain (lyophobic) colloidal systems, Verwey and Overbeek (1948) found that the interatomic potential must fall off faster than r^{-6} at large distances in order for theory and experiment to be consistent. They suggested that at large atomic separations — that is, at separations large compared with atomic transition wavelengths — the London theory must be modified to account for retardation. Such a modification was worked out by Casimir and Polder in 1948. They derived an expression equivalent to (3.85) and showed that, for large r, $V(r) \propto r^{-7}$. The simplest way to obtain the Casimir–Polder result is to argue that for distances large enough for retardation to be important, (3.85) may effectively be replaced by[14]

$$V(r) \cong -\frac{\hbar}{\pi c^6}\alpha_A\alpha_B \int_0^\infty d\omega\omega^6 G\left(\frac{\omega r}{c}\right), \qquad (3.89)$$

[13] See, for instance, P. W. Milonni and J. H. Eberly, *Lasers*, Chapter 3.

[14] This may be justified quantitatively by making the same change in the path of integration in (3.85) as in (3.87). Then it can be seen that the zero-frequency polarizability makes the dominant contribution for $|\omega_{mn}|r/c >> 1$, i.e., when retardation is important. This condition is roughly equivalent to $r >> 137a_o$, where a_o is the Bohr radius.

where α_A, α_B are the static polarizabilities. The integral may be evaluated by introducing a cutoff function $e^{-\lambda\omega r/c}$, $\lambda > 0$, and taking the limit $\lambda \to 0$ after integrating. For instance,

$$\begin{aligned}\int_0^\infty d\omega\omega^6 \frac{\sin 2\omega r/c}{(\omega r/c)^2} &= \frac{c^7}{r^7}\int_0^\infty du u^4 \sin 2u \\ &\to \frac{c^7}{r^7}\lim_{\lambda\to 0}\int_0^\infty du u^4 e^{-\lambda u}\sin 2u = \frac{48c^7}{r^7}\,.\end{aligned} \tag{3.90}$$

We find in this limit, for large r,

$$V(r) \cong -\frac{23\hbar c}{4\pi r^7}\alpha_A\alpha_B, \tag{3.91}$$

which is the Casimir–Polder result.

In either the retarded (Casimir–Polder) or nonretarded (London) limit the van der Waals interaction may be regarded as a consequence of the fluctuating vacuum electromagnetic field. Our derivation leading to (3.82) shows that the van der Waals interaction results from the fact that

$$\langle \text{vac}|E^{(+)}_{o,\mathbf{k}\lambda}(\mathbf{x}_A,t)_i E^{(-)}_{o,\mathbf{k}\lambda}(\mathbf{x}_B,t)_j|\text{vac}\rangle \neq 0. \tag{3.92}$$

In other words, the van der Waals interaction results from *correlations* of the vacuum field over distances on the order of $|\mathbf{x}_A - \mathbf{x}_B| = r$. In more physical terms, the vacuum field induces fluctuating dipole moments in the two atoms, and the dipole–dipole interaction of these zero-mean but correlated moments is the van der Waals interaction.

At the conclusion of their paper Casimir and Polder argued that the simple form of (3.91) might allow it to be derived "by more elementary considerations" than the perturbation-theoretic approach they employed and that "This would be desirable since it would also give a more physical background to our result, a result which in our opinion is rather remarkable. So far we have not been able to find such a simple argument." Not long thereafter Casimir (1949) gave a derivation based on the fluctuating zero-point field, and the derivation given in this section follows closely the spirit of his insightful analysis.[15] The following section is also based on Casimir's work.

[15] See also Boyer (1972,1980) and Renne (1971).

3.12 Force on an Atom near a Conducting Wall

Casimir and Polder (1948) also considered a simpler problem in their study of long-range, retarded interactions; namely, the interaction between an atom and a perfectly conducting wall. For short distances d of the atom from the wall the attractive potential $V(d)$ may be obtained from the dipole–dipoleole interaction of the atom with its image in the wall, and varies as d^{-3}. For large d, however, $V(d)$ falls off as d^{-4}; as in the case of the van der Waals force, the effect of retardation is to weaken the interaction by a factor $\propto d^{-1}$.

Consider first an atom located at the point $\mathbf{R} \equiv (L/2, L/2, d)$ inside the rectangular parallelepiped described by the mode functions (2.91)–(2.93). The energy (3.72) in this case is

$$
\begin{aligned}
-\frac{\alpha}{2}\sum_{\mathbf{k}\lambda}(2\pi\hbar\omega)|\mathbf{A}_{\mathbf{k}\lambda}(\mathbf{R})|^2 = \\
- \frac{4\alpha}{V}\sum_{\mathbf{k}\lambda}(2\pi\hbar\omega)[e^2_{\mathbf{k}\lambda x}\cos^2\frac{1}{2}k_xL\sin^2\frac{1}{2}k_yL\sin^2 k_zd \\
+ e^2_{\mathbf{k}\lambda y}\sin^2\frac{1}{2}k_xL\cos^2\frac{1}{2}k_yL\sin^2 k_zd \\
+ e^2_{\mathbf{k}\lambda z}\sin^2\frac{1}{2}k_xL\sin^2\frac{1}{2}k_yL\cos^2 k_zd] \\
\rightarrow - \left(\frac{2\pi\hbar\alpha}{V}\right)\sum_{\mathbf{k}\lambda}\omega_k[(e^2_{\mathbf{k}\lambda x}+e^2_{\mathbf{k}\lambda y})\sin^2 k_zd + e^2_{\mathbf{k}\lambda z}]\cos^2 k_zd,
\end{aligned}
\tag{3.93}
$$

where, as in the preceding section, we have replaced the polarizability $\alpha(\omega)$ by the static polarizability $\alpha(\omega) \equiv \alpha$, arguing that only the value of $\alpha(\omega)$ at $\omega = 0$ contributes at large distances d of the atom from one of the (conducting) walls of the parallelepiped. In the last expression we have also replaced $\sin^2\frac{1}{2}k_yL, \cos^2\frac{1}{2}k_yL$, etc. by their average value, 1/2. We now define the potential $V(d)$ describing the interaction of the atom with a conducting *wall* as the difference between (3.93) for d finite and for $d \rightarrow \infty$. In the latter limit we replace $\sin^2 k_zd, \cos^2 k_zd$ by 1/2. Thus

$$
V(d) = -\left(\frac{2\pi\hbar\alpha}{V}\right)\sum_{\mathbf{k}\lambda}\omega_k[e^2_{\mathbf{k}\lambda x}+e^2_{\mathbf{k}\lambda y}-e^2_{\mathbf{k}\lambda z}][\sin^2 k_zd-\frac{1}{2}]
$$

$$
\begin{aligned}
&= \left(\frac{\pi\hbar\alpha}{V}\right)\sum_{\mathbf{k}}\omega_k\cos 2k_z d\sum_{\lambda}[e^2_{\mathbf{k}\lambda x}+e^2_{\mathbf{k}\lambda y}-e^2_{\mathbf{k}\lambda z}]\\
&= \left(\frac{\pi\hbar\alpha}{V}\right)\sum_{\mathbf{k}}\omega_k\cos 2k_z d(2k_z^2/k^2)\\
&= \left(\frac{2\pi\hbar\alpha}{V}\right)\frac{V}{8\pi^3}\int d^3k\omega\frac{k_z^2}{k^2}\cos 2k_z d\\
&= \left(\frac{\alpha\hbar c}{2\pi}\right)\int_0^\infty dk k^3\int_0^\pi d\theta\sin\theta\cos^2\theta\cos(2kd\cos\theta)\\
&= \left(\frac{\alpha\hbar c}{\pi}\right)\int_0^\infty dk k^3\left(\frac{\sin 2kd}{2kd}+\frac{2\cos 2kd}{4k^2d^2}-\frac{2\sin 2kd}{8k^3d^3}\right).
\end{aligned}
\tag{3.94}
$$

Evaluating the integral using the procedure exemplified by (3.90), we obtain the Casimir–Polder result

$$V(r) = -\frac{3\alpha\hbar c}{8\pi d^4}. \tag{3.95}$$

In Chapter 8 we discuss experimental evidence for the Casimir–Polder force.

3.13 The Magnetic Moment of the Electron

In order to explain the spectra of atoms in magnetic fields, Uhlenbeck and Goudsmit (1926) postulated that the electron has an intrinsic (spin) angular momentum $\hbar/2$ and a magnetic dipole moment $e\hbar/2mc \equiv \mu_o$, the Bohr magneton. Both properties of the electron were later found by Dirac (1928) to be consequences of relativistically invariant quantum mechanics.

Recall that a curent loop enclosing a plane area A has a magnetic dipole moment $\mu = IA/c$, where I is the current. For a charge e moving in a circular orbit of radius r, $\mu = (\pi r^2)(e\nu)/c = (e/2mc)L$, where ν and L are, respectively, the orbital frequency and angular momentum. Therefore the gyromagnetic ratio $\mu/L = e/2mc$. For the electron magnetic dipole moment and spin angular momentum, however, $\mu_s/L_s = \mu_o/(\hbar/2) = 2(e/2mc)$. That is, the Landé g-factor for electron spin is 2, as predicted by the Dirac theory without coupling of the electron to the radiation field.

As in the case of the Lamb shift, radiative corrections give small departures from this prediction. Just prior to the first accurate measurements by Kusch et al. (see Kusch and Foley, 1948), Schwinger (1948) calculated for the "anomaly" $(g-2)/2$ the value $\alpha/2\pi \cong .00116$; the experimentalists reported a value $.00119 \pm .00005$.

And like the Lamb shift, the anomalous moment of the electron provides one of the most sensitive tests of QED. Recent experiments by Dehmelt et al.[16] give a value of $(g-2)/2$ more accurate than all previous measurements by a factor of nearly 1000:

$$\frac{g-2}{2} = .001159652188(4) \,. \tag{3.96}$$

A QED calculation up to fourth order in the fine–structure constant α yields (Kinoshita, 1989)

$$\frac{g-2}{2} = .001159652192(74) \,. \tag{3.97}$$

Here the theoretical "error" is due mainly to the uncertainty in the fine-structure constant. Such a comparison of theory and experiment explains the cliché that QED is "the best theory we have!"

As in our discussion of the Lamb shift, we will focus our attention here on the nonrelativistic theory of the anomalous moment of the electron. state

From the interaction Hamiltonian $-(e\hbar/2mc)\boldsymbol{\sigma}\cdot\mathbf{B}$ describing the coupling of electron spin to a magnetic field $\mathbf{B}$, and the commutation relations for the Pauli spin—1/2 operators $\sigma_x, \sigma_y, \sigma_z$, we obtain the Heisenberg equation of motion

$$\frac{d\boldsymbol{\sigma}}{dt} = \frac{e}{mc}\mathbf{B}\times\boldsymbol{\sigma} \tag{3.98}$$

for $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. If $\mathbf{B} = 0, \boldsymbol{\sigma}$ is constant in time. However, in QED $\mathbf{B}$ is an operator which, like $\mathbf{E}$, has zero expectation value but nonvanishing variance in the vacuum state of the field. Thus, for an electron in free space,

$$\begin{aligned}\frac{d\boldsymbol{\sigma}}{dt} &= \frac{ie}{mc}\sum_{\mathbf{k}\lambda}\left(\frac{2\pi\hbar c^2}{\omega_k V}\right)^{1/2}[a_{\mathbf{k}\lambda}(0)e^{-i\omega_k t}e^{i\mathbf{k}\cdot\mathbf{r}} - a^{\dagger}_{\mathbf{k}\lambda}(0)e^{i\omega_k t}e^{-i\mathbf{k}\cdot\mathbf{r}}] \\ &\quad\times(\mathbf{k}\times\mathbf{e}_{\mathbf{k}\lambda})\times\boldsymbol{\sigma}(t) \,,\end{aligned} \tag{3.99}$$

where we have used equation (2.57). In writing this equation we are ignoring the part of the magnetic field that depends on $\boldsymbol{\sigma}$. That is, we are including the effect of the vacuum $\mathbf{B}$ field, but not the radiation reaction $\mathbf{B}$ field.

In the lowest order of approximation we use $\boldsymbol{\sigma}(t) \cong \boldsymbol{\sigma}(0)$, the zero-coupling solution, on the right side of (3.99). Then

$$\boldsymbol{\sigma}(t) \cong \boldsymbol{\sigma}(0) - \frac{e}{mc}\sum_{\mathbf{k}\lambda}\left(\frac{2\pi\hbar c^2}{\omega_k V}\right)^{1/2}\frac{1}{\omega_k}[a_{\mathbf{k}\lambda}(0)e^{-i\omega_k t}e^{i\mathbf{k}\cdot\mathbf{r}}$$

[16] See Dehmelt (1990) and references therein, and Chapter 6, where the experiments are briefly described.

$$+ a^{\dagger}_{\mathbf{k}\lambda}(0)e^{i\omega_k t}e^{-i\mathbf{k}\cdot\mathbf{r}}](\mathbf{k}\times\mathbf{e}_{\mathbf{k}\lambda})\times\boldsymbol{\sigma}(0)\ , \tag{3.100}$$

where in the second term we ignore a contribution from the artificial switch-on at $t = 0$ of the interaction. Using the vacuum expectation values $\langle a_{\mathbf{k}\lambda}(0)a_{\mathbf{k}'\lambda'}(0)\rangle = \langle a^{\dagger}_{\mathbf{k}\lambda}(0)a_{\mathbf{k}'\lambda'}(0)\rangle = 0$ and $\langle a_{\mathbf{k}\lambda}a^{\dagger}_{\mathbf{k}'\lambda'}(0)\rangle = \delta^3_{\mathbf{k},\mathbf{k}'}\delta_{\lambda\lambda'}$, we obtain

$$\langle\Delta\boldsymbol{\sigma}^2\rangle \cong \langle\left(\frac{e}{mc}\right)^2\sum_{\mathbf{k}}\left(\frac{2\pi\hbar c^2}{\omega_k V}\right)\frac{1}{\omega_k^2}\sum_{\lambda}[(\mathbf{k}\times\mathbf{e}_{\mathbf{k}\lambda})\times\boldsymbol{\sigma}(0)]^2\rangle\ , \tag{3.101}$$

with $\Delta\boldsymbol{\sigma} \equiv \boldsymbol{\sigma}(t) - \boldsymbol{\sigma}(0)$. Now

$$\begin{aligned}\sum_{\lambda}[(\mathbf{k}\times\mathbf{e}_{\mathbf{k}\lambda})\times\boldsymbol{\sigma}(0)]^2 &= \sum_{\lambda}[\mathbf{k}(\boldsymbol{\sigma}(0)\cdot\mathbf{e}_{\mathbf{k}\lambda}) - \mathbf{e}_{\mathbf{k}\lambda}(\mathbf{k}\cdot\boldsymbol{\sigma}(0))]^2\\ &= k^2\sum_{\lambda}(\boldsymbol{\sigma}(0)\cdot\mathbf{e}_{\mathbf{k}\lambda})^2 + (\mathbf{k}\cdot\boldsymbol{\sigma}(0))^2\sum_{\lambda}(1)\\ &= k^2\boldsymbol{\sigma}^2(0) - (\mathbf{k}\cdot\boldsymbol{\sigma}(0))^2 + 2(\mathbf{k}\cdot\boldsymbol{\sigma}(0))^2\\ &= k^2\boldsymbol{\sigma}^2(0) + (\mathbf{k}\cdot\boldsymbol{\sigma}(0))^2\ ,\end{aligned} \tag{3.102}$$

and the integration over all solid angles about $\mathbf{k}$ of this expression is

$$\int d\Omega_{\mathbf{k}}[k^2 + (\mathbf{k}\cdot\boldsymbol{\sigma}(0))^2] = 4\pi k^2\boldsymbol{\sigma}^2(0) + \frac{4\pi}{3}k^2\boldsymbol{\sigma}^2(0) = \frac{16\pi}{3}k^2\boldsymbol{\sigma}^2(0)\ . \tag{3.103}$$

The replacement $\sum_{\mathbf{k}} \to (V/8\pi^3)\int dk k^2\int d\Omega_{\mathbf{k}}$ in (3.101) then yields

$$\frac{\langle\Delta\boldsymbol{\sigma}^2\rangle}{\langle\boldsymbol{\sigma}^2(0)\rangle} \cong \frac{4\alpha}{3\pi}\left(\frac{\hbar}{mc}\right)^2\int dk k \to \frac{2\alpha}{3\pi}\left(\frac{\hbar K}{mc}\right)^2 \tag{3.104}$$

when, in this nonrelativistic approach, we introduce an upper limit K in the integration over $k = \omega/c$ in order to avoid a divergence.

In the absence of any coupling to the vacuum magnetic field, the electron spin has a fixed direction. Following Welton (1948), we define the mean-square fluctuation angle

$$\langle\Delta\theta^2\rangle \equiv \frac{\langle\Delta\boldsymbol{\sigma}^2\rangle}{\langle\boldsymbol{\sigma}^2(0)\rangle} = \frac{2\alpha}{3\pi}\left(\frac{\hbar K}{mc}\right)^2 \tag{3.105}$$

and consider the expectation value

$$-\frac{e\hbar}{2mc}\langle\boldsymbol{\sigma}\rangle\cdot\mathbf{B}_{\text{ext}} = -\frac{e\hbar}{2mc}|\langle\boldsymbol{\sigma}\rangle|B_{\text{ext}}\cos\theta \tag{3.106}$$

of the potential energy of the spin in an *external* magnetic field, where θ is the angle between $\boldsymbol{\sigma}$ and the direction of $\mathbf{B}_{\text{ext}}$. The effect of the fluctuating vacuum **B** field is to replace $\cos\theta$ by

$$\langle\cos(\theta+\Delta\theta)\rangle \cong \cos\theta\left[1-\frac{1}{2}\langle\Delta\theta^2\rangle\right] = \cos\theta\left[1-\frac{\alpha}{3\pi}\left(\frac{\hbar K}{mc}\right)^2\right], \quad (3.107)$$

so that

$$-\frac{e\hbar}{2mc}\langle\boldsymbol{\sigma}\rangle\cdot\mathbf{B}_{\text{ext}} \to -\frac{e\hbar}{2mc}|\langle\boldsymbol{\sigma}\rangle|B_{\text{ext}}\left[1-\frac{\alpha}{3\pi}\left(\frac{\hbar K}{mc}\right)^2\right], \quad (3.108)$$

or, in effect,

$$\frac{e\hbar}{2mc} \to \frac{e\hbar}{2mc}\left[1-\frac{\alpha}{3\pi}\left(\frac{\hbar K}{mc}\right)^2\right]. \quad (3.109)$$

This implies

$$\frac{g-2}{2} = -\frac{\alpha}{3\pi}\left(\frac{\hbar K}{mc}\right)^2 \quad (3.110)$$

to first order in α.

The problem with this result is that it has the wrong sign: experiment shows that $(g-2)/2$ is *positive*. We can rectify this situation starting from the observation that radiation reaction has been ignored. Although it turns out that radiation reaction does not affect the potential energy $-(e\hbar/2mc)\boldsymbol{\sigma}\cdot\mathbf{B}$ to first order in α, it does contribute to the electron mass at this order [Equation (3.33)]. Since we have left out radiation reaction in the calculation leading to (3.109), the mass in that expression must actually be the *bare* mass m_o. What is measured experimentally, of course, involves the observed mass $m = m_o + \delta m$. Therefore we should express (3.109) in terms of the observed mass. This is accomplished by the replacement

$$\begin{aligned}
\frac{e\hbar}{2m_oc}\left[1-\frac{\alpha}{3\pi}\left(\frac{\hbar K}{mc}\right)^2\right] &\to \frac{e\hbar}{2mc}\left(\frac{m_o+\delta m}{m_o}\right)\left[1-\frac{\alpha}{3\pi}\left(\frac{\hbar K}{mc}\right)^2\right] \\
&\cong \frac{e\hbar}{2mc}\left[1-\frac{\alpha}{3\pi}\left(\frac{\hbar K}{mc}\right)^2\right]\left[1+\frac{\delta m}{m}\right] \\
&= \frac{e\hbar}{2mc}\left[1-\frac{\alpha}{3\pi}\left(\frac{\hbar K}{mc}\right)^2\right]\left[1+\frac{4\alpha}{3\pi mc^2}(\hbar Kc)\right] \\
&\cong \frac{e\hbar}{2mc}\left[1+\frac{4\alpha}{3\pi}\left(\frac{\hbar K}{mc}\right)-\frac{\alpha}{3\pi}\left(\frac{\hbar K}{mc}\right)^2\right],
\end{aligned} \quad (3.111)$$

where we have used (3.33) with a cutoff $E = \hbar K c$ in the upper limit of integration. This implies

$$\frac{g-2}{2} \cong \frac{4\alpha}{3\pi}\left(\frac{\hbar K}{mc}\right) - \frac{\alpha}{3\pi}\left(\frac{\hbar K}{mc}\right)^2 \tag{3.112}$$

which is positive for any cutoff $K < 4mc/\hbar$. Thus the predicted anomaly in the nonrelativistic theory is positive for all cutoffs for which the nonrelativistic theory is sensible. The choice $K = 0.42mc/\hbar$ yields $(g-2)/2 = \alpha/2\pi$, which is the relativistic QED result to first order in α (Grotch and Kazes, 1977).

We conclude therefore that, as in spontaneous emission, both vacuum field fluctuations and radiation reaction are important for the anomalous magnetic moment of the electron (Grotch and Kazes, 1977; Dupont-Roc, Fabre, and Cohen-Tannoudji, 1978).

However, the reader is warned not to take these calculations too seriously, for the result $(g-2)/2 = \alpha/2\pi$ could be obtained by retaining only the first (radiation reaction) term in (3.112) and choosing $K = 3mc/8\hbar$. It should also be noted that the solution $K \cong 0.42mc/\hbar$ of (3.112) with $(g-2)/2 = \alpha/2\pi$ is not unique.

3.14 Summary

We have shown in this chapter how some basic QED effects may be understood physically as consequences of the fluctuating vacuum electromagnetic field. These effects include such commonplace phenomena as spontaneous emission and van der Waals forces and also the Lamb shift and the anomalous moment of the electron, which provide the most important tests of QED. Consideration of these vacuum effects leads us to the concept of renormalization as a means of obtaining finite results from otherwise infinite quantitites. Vacuum fluctuations and renormalization are two of the most important features of modern physics.

It is hoped that this chapter has convinced (or reminded) the reader that the vacuum — or the electromagnetic vacuum, at least — is a quantum state with observable physical consequences.

These physical explanations of various QED vacuum effects have considerable esthetic appeal and seem to offer compelling evidence for the "reality" of vacuum field fluctuations. And yet the vacuum field fluctuations are not the only physical basis for understanding these phenomena. There is another basis — *source* fields — upon which we can construct physical interpretations of QED vacuum effects. This point is pursued further in the following chapters.

3.15 Bibliography

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