Derivation of the springback equation

In order to derive the springback equation, we start by considering a straight beam that, by applying a bending force, is bent to a certain radius R as shown in figure 1.

Now, for the following calculations we consider that: i) the material obeys Hooke's law (linear elastic material), ii) the Young modulus, E, is the same for tension and compression and iii) the transverse fibers will remain plane after bending.

On a straight beam, we have that the distance $\overline{AB}$ is the same as $\overline{CD}$. Now, as shown in figure 1, the distance $\overline{AB}$ remains unchanged after bending because this corresponds to the neutral surface which does not experience strain. The distance $\overline{C'D'}$ corresponds to the distance $\overline{CD}$ under a tensile force (if it would be on the opposite side of the neutral surface it would be under a compressive force). The longitudinal strain due to bending, ε_b, of a surface that is at a y distance from the neutral surface can be calculated by:

$$\varepsilon_b = \frac{\overline{C'D'} - \overline{CD}}{\overline{CD}} \tag{1}$$

Before bending we have that $\overline{AB} = \overline{CD} = R\theta$. Then, after bending, the arc-length of $\overline{C'D'}$ is given by $\overline{C'D'} = (R+y)\theta$. By plugging these values in equation (1) we get:

$$\varepsilon_b = \frac{(R+y)\theta - R\theta}{R\theta} \Leftrightarrow \varepsilon_b = \frac{y}{R} \tag{2}$$

The longitudinal stress developed by bending is given by $\sigma_b = E\varepsilon_b$, which by using equation (2) can now be defined as:

$$\sigma_b = \frac{Ey}{R} \tag{3}$$

Let's now consider a cross-section of a generic beam as shown in figure 2 where it is shown the relationship between the longitudinal stress and the bending moment.

The longitudinal bending stress distribution, σ_b, across the yy axis is displayed on the right side of the beam. The distribution is such that it is zero at the neutral axis and follows a linear equation with slope $\pm E/R$. The force dF associated to an element of area dA (shaded area in figure 2) can be determined by $dF = \sigma_b dA$. The total force, F_T, can now be calculated by:

$$F_T = \int dF = \int \sigma_b dA \tag{4}$$

To calculate the bending moment, M_b, we need to multiply the force, F_T, by the distance, y, to the neutral surface of the beam, which leads to:

$$M_b = F_T\, y = \int y\, \sigma_b\, dA = \frac{E}{R} \int y^2 dA \tag{5}$$

The part of the integral defined as $\int y^2 dA$ corresponds to the moment of inertia of the beam, J, which leads to a simplified version of the bending moment expression:

$$M_b = \frac{EJ}{R} \tag{6}$$

A beam bent to a radius R_b will develop a bending moment M_b. If we overcome the yield stress of the material, after unloading the beam, the elastic energy will be released and the final radius will be a bigger one, R_f. We can define curvature as the inverse function of the radius, $1/R$ and then, if we plot the loading and the unloading of the beam (assuming that the beam undergoes plastic deformation), we will get something similar to the plot in figure 3.

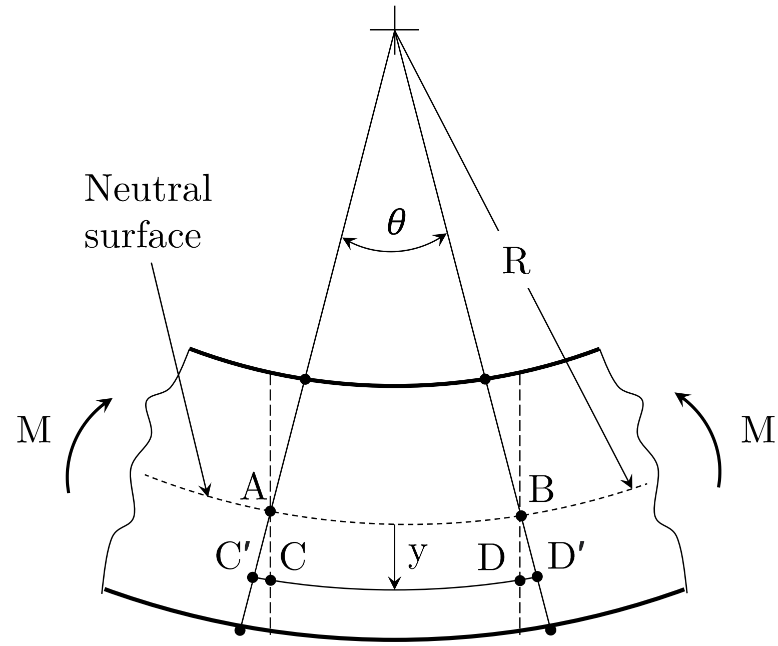

Figure 1: Strain development in beam bending.

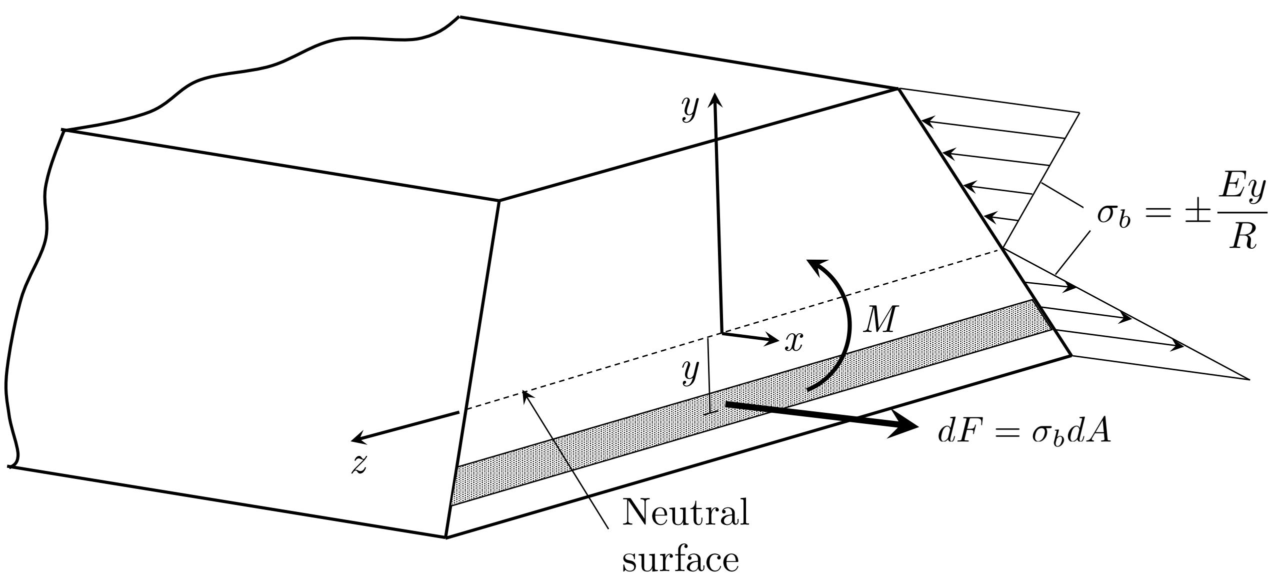

Figure 2: Relationship between the longitudinal stress and the bending moment.

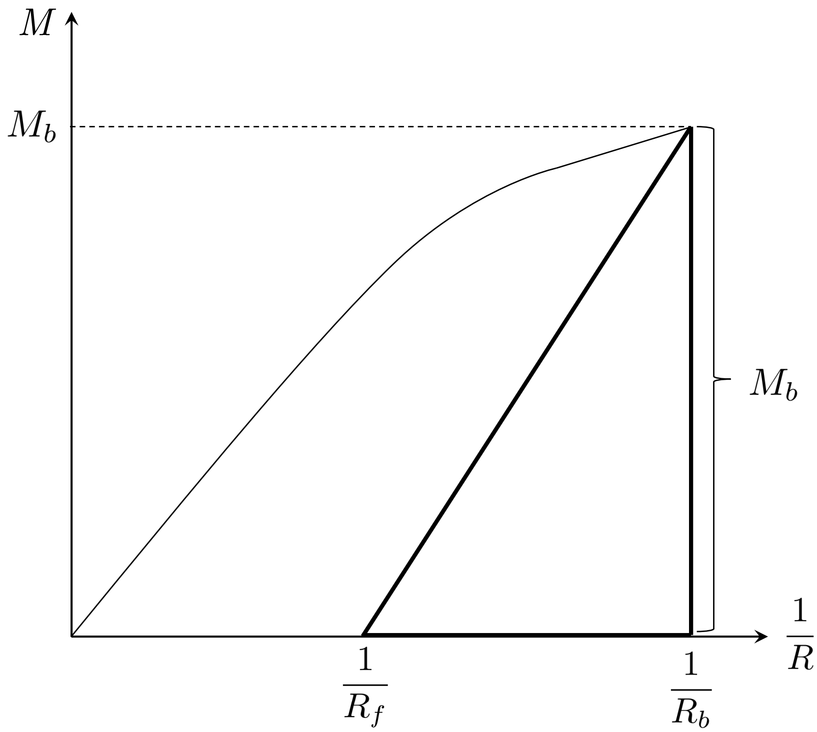

Figure 3: Bending moment versus curvature of a bent beam.

From the schematic plot we see that after developing the bending moment M_b and undergoing plastic deformation, we reach a curvature of $1/R_b$. Then, after unloading, the elastic energy is released and the curvature becomes $1/R_f$, which is the final radius of the beam.

The ratio $M_b/(1/R_b - 1/R_f)$ corresponds to the slope of the unloading line, which then corresponds to the elastic part of the whole bending process. The slope can be mathematically defined, in this case, by $\partial M_b/\partial(1/R)$, and we can then write that:

$$\frac{\partial M_b}{\partial(1/R)} = \frac{M_b - 0}{(1/R_b) - (1/R_f)}$$

Which can be re-written as:

$$\frac{1}{R_b} - \frac{1}{R_f} = \frac{M_b}{\partial M_b/\partial(1/R)} \tag{7}$$

From equation (6), we see that M_b is a function of the curvature $1/R$, which means that the partial derivative that defines the slope is:

$$\frac{\partial M_b}{\partial(1/R)} = EJ\frac{\partial}{\partial(1/R)}\left(\frac{1}{R}\right) = EJ \tag{8}$$

Resulting finally in the equation that characterizes the springback effect:

$$\frac{1}{R_b} - \frac{1}{R_f} = \frac{M_b}{EJ} \qquad \blacksquare$$