These notes closely follow the presentation of the material given in David C. Lay's textbook Linear Algebra and its Applications (3rd edition). These notes are intended primarily for in-class presentation and should not be regarded as a substitute for thoroughly reading the textbook itself and working through the exercises therein.

Linearly Independent Sets and Linearly Dependent Sets

Definition *An indexed set of vectors* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *in a vector space* V *is said to be* **linearly independent** *if the vector equation*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

has only the trivial solution ($c_1 = c_2 = \cdots = c_k = 0$*).*

If the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *is not linearly independent, then it is said to be* **linearly dependent**.

If the of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent, then there exist scalars, $c_1, c_2, \ldots, c_k$, at least one of which is not equal to 0, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

An equation such as the above (with not all of the scalar coefficient equal to 0) is called a **linear dependence relation**.

Example *Let* $f : \Re \to \Re$ *be the function* $f(x) = sin(x)$ *and let* $g : \Re \to \Re$ *be the function* $g(x) = cos(x)$*. Then, f and g are "vectors" in the vector space* $C^0(\Re)$*. Is the set of functions* $\{f, g\}$ *linearly independent or linearly dependent? If this set is linearly dependent, then give a linear dependence relation for the set.*

Example *Let $f : \Re \rightarrow \Re$ be the function $f(x) = 9\,sin(2x)$ and let $g : \Re \rightarrow \Re$ be the function $g(x) = 8\,sin(x)\,cos(x)$. Then, f and g are "vectors" in the vector space $C^0(\Re)$. Is the set of functions $\{f, g\}$ linearly independent or linearly dependent? If this set is linearly dependent, then give a linear dependence relation for the set.*

Example *Let p_1, p_2, and p_3 be the polynomial functions (with domain $\Re$) defined by*

$$p_1(t) = -3t^2 + 5t + 3$$
$$p_2(t) = 12t^2 + 4t - 18$$
$$p_3(t) = 6t^2 - 2t - 8.$$

These functions are "vectors" in the vector space $P_2(\Re)$. Is the set of vectors $\{p_1, p_2, p_3\}$ linearly independent or linearly dependent? If this set is linearly dependent, then give a linear dependence relation for the set.

Solution *We need to consider the vector equation*

$$c_1 p_1 + c_2 p_2 + c_3 p_3 = z$$

where z is the zero vector of $P_2(\Re)$. (In other words, $z : \Re \to \Re$ is the function defined by $z(t) = 0$ for all $t \in \Re$.) The above equation is equivalent to

$$c_1 p_1(t) + c_2 p_2(t) + c_3 p_3(t) = z(t) \text{ for } \textbf{all } t \in \Re$$

or

$$c_1(-3t^2 + 5t + 3) + c_2(12t^2 + 4t - 18) + c_3(6t^2 - 2t - 8) = 0 \text{ for } \textbf{all } t \in \Re.$$

Rearrangement of the above equation gives

$$(-3c_1 + 12c_2 + 6c_3)t^2 + (5c_1 + 4c_2 - 2c_3)t + (3c_1 - 18c_2 - 8c_3) = 0 \text{ for } \textbf{all } t \in \Re.$$

The above equation states that a certain quadratic function is equal to the zero function. This can be true only if all of the coefficients of this quadratic function are equal to zero. Therefore, we must have

$$-3c_1 + 12c_2 + 6c_3 = 0$$
$$5c_1 + 4c_2 - 2c_3 = 0$$
$$3c_1 - 18c_2 - 8c_3 = 0.$$

Solving this linear system in the usual way,

$$\begin{bmatrix} -3 & 12 & 6 & 0 \\ 5 & 4 & -2 & 0 \\ 3 & -18 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

we see that there are non–trivial solutions. One such solution is

$$c_1 = 2$$
$$c_2 = -1$$
$$c_3 = 3.$$

Our conclusion is that the set of functions $\{p_1, p_2, p_3\}$ is linearly dependent and that a linear dependence relation for this set is

$$2p_1 - p_2 + 3p_3 = z.$$

Theorem *An indexed set,* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$*, of two or more vectors, with* $\mathbf{v}_1 \neq \mathbf{0}$*, is linearly dependent if and only if some* $\mathbf{v}_j$ *(with* $j > 1$*) is a linear combination of the preceding vectors,* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}$.

Bases

Definition *A* **basis** *for a vector space, V, is a set of vectors in V that is linearly independent and spans V.*

Example *The set of vectors* $\{\mathbf{e}_1, \mathbf{e}_2\}$ *where*

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \textit{and} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

is a basis for $\Re^2$.

Example *Is the set of vectors* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *where*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 6 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

a basis for $\Re^3$*?*

Solution *Observe that*

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 6 & 0 \\ -3 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

By the Square Matrix Theorem, we conclude that the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *is linearly independent and that this set of vectors also spans* $\Re^3$*. Thus, this set of vectors is a basis for* $\Re^3$*.*

The above example suggests a theorem that follows immediately from the Square Matrix Theorem:

Theorem *If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set (consisting of exactly n vectors) in $\Re^n$, then this set of vectors is a basis for $\Re^n$.*

Also, if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a set (consisting of exactly n vectors) in $\Re^n$ and this set of vectors spans $\Re^n$, then this set of vectors is a basis for $\Re^n$.

What if we have a set of vectors in $\Re^n$ that consists of fewer or more than n vectors? Can such a set possibly be a basis for $\Re^n$?

The answer is "No".

If we have a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of k vectors in $\Re^n$ where $k < n$, then the matrix

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \end{bmatrix}$$

has more rows than columns. Thus, not every row of A can be a pivot row, and it is not possible for the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ to span $\Re^n$.

If we have a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of k vectors in $\Re^n$ where $k > n$, then the matrix

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \end{bmatrix}$$

has more columns than rows. Thus, not every column of A can be a pivot column, and it is not possible for the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ to be linearly independent. (Another way of looking at this is that the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ contains more vectors than there are entries in each vector, so the set must be linearly dependent.)

From the above considerations, we conclude:

Theorem *Every basis for $\Re^n$ contains exactly n vectors.*

Note that the above theorem does **not** say that any set of n vectors in $\Re^n$ is a basis for $\Re^n$. It is certainly possible that a set of n vectors in $\Re^n$ might be linearly dependent. However, any linearly independent set of vectors in $\Re^n$ must also span $\Re^n$ and, conversely, and set of n vectors that spans $\Re^n$ must also be linearly independent.

Example *Let p_1, p_2, and p_3 be the polynomial functions (with domain $\Re$) defined by*

$$p_1(t) = -3t^2 + 5t + 3$$
$$p_2(t) = 12t^2 + 4t - 18$$
$$p_3(t) = 6t^2 - 2t.$$

Is the set of vectors $\{p_1, p_2, p_3\}$ a basis for P_2? If so, write the function $p \in P_2$ defined by

$$p(t) = t^2 - 6t + 7$$

as a linear combination of the functions in the set $\{p_1, p_2, p_3\}$.

Solution *In order to solve the vector equation*

$$c_1 p_1 + c_2 p_2 + c_3 p_3 = p,$$

we need to consider the augmented matrix

$$\begin{bmatrix} -3 & 5 & 3 & 1 \\ 12 & 4 & -18 & -6 \\ 6 & -2 & 0 & 7 \end{bmatrix}$$

which is equivalent to

$$\begin{bmatrix} 1 & 0 & 0 & \frac{119}{96} \\ 0 & 1 & 0 & \frac{7}{32} \\ 0 & 0 & 1 & \frac{29}{24} \end{bmatrix}.$$

We observe that the set of vectors $\{p_1, p_2, p_3\}$ is a basis for P_2 and that

$$\frac{119}{96} p_1 + \frac{7}{32} p_2 + \frac{29}{24} p_3 = p.$$

Bases for the Nullspace and the Column Space of a Matrix

Example *Find bases for the nullspace and the column space of the matrix*

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}.$$

Use the fact that

$$A \sim \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Theorem *The pivot columns of a matrix, A, form a basis for the column space of A.*

Note: It is the pivot columns of A, not the pivot columns of $rref(A)$, that form a basis for the column space of A!