Proving limit laws using infinitesimals

Following we will try to prove the limit laws, that is: The addition Law, Subtraction Law, Constant Law, Multiplication Law, Division Law, Power Law etc. using infinitesimals. In other words, we will first define the limit of a function in terms of infinitesimals, and after that we will use that definition and the definition of the infinitesimals and their properties to prove the above limit laws.

First let's look at the definition and the properties of infinitesimals, for they will be crucial in proving limit laws.

Infinitesimals

Following i will briefly look upon some definitions and properties of infinitesimals that will be crucial in making my point.

Definition1. f(x) is said to be an infinitesimal as $x \to a$,($x \to \infty$) if and only if:

$\forall \epsilon > 0, \exists \delta(\epsilon), or\ (M(\epsilon) > 0)$, such that whenever $0 < |x-a| < \delta$,($|x| > M)$, we have : $|f(x)| < \epsilon$.

Throughout this portion i will denote the infinitesimals by one of the following symbols and their indexes: $\alpha(x),\ \beta(x),\ \ \gamma(x),\ \ \ etc$

Theorem 1. If $\alpha_1(x), \alpha_2(x), \ldots.., \alpha_n(x)$are infinitesimals as $x \to a$then also every linear combination of them is an infinitesimal. That is:
$\alpha(x) = c_1\alpha_1(x) + c_2\alpha_2(x) + \ldots.. + c_n\alpha_n(x)$is an infinitesimal.
The constants are from the field of real numbers.

Proof:
We will only prove this using the definition of infinitesimals,-written above- without relying at all on the definition and the properties of limits.

To show that $\alpha(x)$is an infinitesimal, we need to show that:
$\forall \epsilon > 0, \exists \delta(\epsilon) > 0$such that $|\alpha(x)| < \epsilon$whenever $0 < |x-a| < \delta \quad ?????$

Now since $\alpha_k(x)$where $k = 1, 2, 3, 4, \ldots.., n$, are all infinitesimals, from the definition we have:

$\forall \epsilon > 0$and also for $\frac{\epsilon}{n|c|} > 0, \exists \delta_k(\epsilon) > 0$ such that: $|\alpha_k(x)| < \frac{\epsilon}{|c|n}$, whenever $0 < |x-a| < \delta_k$
.

Now let $\delta = min(\delta_1, \delta_2, \ldots., \delta_n)$, then
$\forall \frac{\epsilon}{|c|n} > 0,\ \exists \delta(\epsilon) > 0, such\ that\ |\alpha_k| < \frac{\epsilon}{n|c|},\ wherever\ \ 0 < |x-a| < \delta$

Now let $\delta = min(\delta_1, \delta_2, \ldots., \delta_n)$, then
$\forall \frac{\epsilon}{|c|n} > 0,\ \exists \delta(\epsilon) > 0, such\ that\ |\alpha_k| < \frac{\epsilon}{n|c|},\ wherever\ \ 0 < |x-a| < \delta$
Now let also $c = max(c_1, c_2, \ldots., c_n)$so,

Now:

$|\alpha(x)| = |c_1\alpha_1(x)+c_2\alpha_2(x)+...+c_n\alpha_n(x)| \leq |c_1||\alpha_1(x)|+|c_2||\alpha_2(x)|+...+|c_n||\alpha_n(x)|<$

$|c|(|\alpha_1(x)|+|\alpha_2(x)|+...+|\alpha_n(x)|) < |c|(\frac{\epsilon}{|c|n}+\frac{\epsilon}{|c|n}+...+\frac{\epsilon}{|c|n})$, so since there are n-terms we get
$|\alpha(x)| < |c|n\frac{\epsilon}{n|c|} = \epsilon.$
Which indeed is what we needed to prove.

Theorem 2. If $\alpha(x)$is an infinitesimal when $x \longrightarrow a$ and g(x) is a bounded function at a, then the product $\alpha(x) * g(x)$is an infinitesimal also.

Proof:
Here we need to show that $|\alpha(x)g(x)| < \epsilon$??

Since g(x) is a bounded function at the point a, it means that there exists a constant K, and also a δ_n neighborhood of a, such that for any $|x-a| < \delta_o, \ \ we \ \ have \ \ |g(x)| \leq K.$

Now, since $\alpha(x)$is an infinitesimal it means that:
$\forall\epsilon > 0, also \ \ for \ \ \frac{\epsilon}{K} > 0, \exists\delta_1 > 0, such \ \ that \ \ |\alpha(x)| < \frac{\epsilon}{K}, \ \ whenever \ \ 0 < |x-a| < \delta_1.$

Now if we let, $\delta = min(\delta_o, \delta_1), we \ \ get$

$\forall\epsilon > 0, also \ \ for \ \ \frac{\epsilon}{K} > 0, \exists\delta_1 > 0, such \ \ that \ \ |\alpha(x)| < \frac{\epsilon}{K}, \ \ whenever \ \ 0 < |x-a| < \delta.$

Now from the above we have:

$|g(x)\alpha(x)| = |g(x)||\alpha(x)| < K\frac{\epsilon}{K} = \epsilon.$

Which indeed is what we actually wanted to prove.

Corollary 1. The product of an infinitesimal with a function that has a limit as $x \longrightarrow a$is an infinitesimal at that point.

Proof: The proof is quite straightforward, once we notice that every function that has a limit at any limit point, say a, is actually also bounded at that point. So, we go back on the conditions of Theorem 1.

Corollary 2. The product of two or more infinitesimals as x->a, is an infinitesimal at that point. That is :
$\alpha(x) = \alpha_1(x) * \alpha_2(x) * ...... * \alpha_n(x)$

Proof: Since $\alpha_k(x), \ \ for \ \ k = 1,2,3, ..., n$are infinitesimals as x-->a, we know that:

$\forall\epsilon > 0, \ \ also \ \ for \ \sqrt[n]{\epsilon}, \exists\delta_k > 0, \ \ such \ \ that \ |\alpha_k(x)| < \sqrt[n]{(\epsilon)}, \ \ whenever \ \ 0 < |x-a| < \delta_k.$

Now let $\delta = min(\delta_1, \delta_2, ..., \delta_n)$ then we get

$$\forall \epsilon > 0, \ \ also \ for \ \sqrt[n]{\epsilon}, \exists \delta > 0, \ \ such \ \ that \ |\alpha_k(x)| < \sqrt[n]{(\epsilon)}, \ \ whenever \ \ 0 < |x{-}a| < \delta.$$

To prove the theorem, we actually need to show that $|\alpha(x)| < \epsilon$??

So,

$$|\alpha(x)| = |\alpha_1(x)*\alpha_2(x)*.......*\alpha_n(x)| = |\alpha_1(x)|*|\alpha_2(x)|*....*|\alpha_n(x)| < \sqrt[n]{\epsilon}*\sqrt[n]{\epsilon}*\sqrt[n]{\epsilon}*....*\sqrt[n]{\epsilon} = (\sqrt[n]{\epsilon})^n = \epsilon.$$

Corollary 3. The product of any infinitesimal when x-->a, with a constant C, is again an infinitesimal at this point.

Proof: The proof of this is trivial.

Corollary 4. If $\alpha(x)$ is an infinitesimal when x->a, and $\lim_{x \to a} g(x) = b = / = 0, \ \ then \ \frac{\alpha(x)}{g(x)}$ is also an infinitesimal.

Proof:

We need to show that

$$\forall \epsilon > 0, \exists \delta > 0, \ \ such \ \ that \ \ |\frac{\alpha(x)}{g(x)}| < 0 \ \ ??$$

Since $\alpha(x)$ is an infinitesimal we know that:

$$\forall \epsilon > 0, \ \ also \ for \ \frac{\epsilon}{M} > 0, \exists \delta_1 > 0, \ \ such \ that \ |\alpha(x)| < \frac{\epsilon}{M}, \ \ wherever \ \ 0 < |x{-}a| < \delta_1.$$

Also, since $\lim_{x \to a} g(x) = b$ from a theorem, which I am not going to prove here, we know that:

$\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{b}$. From here, since $\frac{1}{g(x)}$ has a limit at the limit point a, it means that it is also bounded at that point.
In other words,
$\exists \ M > 0, \ \ and \ also \ a \ \delta_0 > 0 \ neighborhood \ of \ \ a$ such that whenever $|x - a| < \delta_0$ we have $\frac{1}{|g(x)|} \leq M.$

Now let $\delta = min(\delta_0, \delta_1)$, so now we have :

$$\forall \epsilon > 0, \ \ also \ \ for \ \frac{\epsilon}{M} > 0, \exists \delta > 0, \ \ such \ that \ |\alpha(x)| < \frac{\epsilon}{M}, \ \ and \ \frac{1}{|g(x)|} \leq M \ \ wherever \ \ 0 < |x{-}a| < \delta.$$

Now:

$$|\frac{\alpha(x)}{g(x)}| = \frac{|\alpha(x)|}{|g(x)|} \leq M * |\alpha(x)| < M * \frac{\epsilon}{M} = \epsilon.$$

So, this way we proved that $\frac{\alpha(x)}{g(x)}$ is also an infinitesimal.

The following theorem is the crucial one on proving the limit laws. Moreover, i will try to use this theorem as a definition of the limit of a function, and hence use this definition to prove the limit laws.

THEOREM (*) ***A is said to be the limit of a function f(x) as x-->a, if and only if:*** $f(x) - A = \alpha(x)$***. In other words, if an only if the difference*** $f(x) - A$***, is an infinitesimal at the point a.***

Proof: Let's first suppose that: $\lim x \to af(x) = A$, we have to show that $f(x) - A = \alpha(x)???$

From the Cauchy's definition of the limit of a function we have:

$$\forall\epsilon > 0, \exists\delta(\epsilon) > 0,\ \ such\ that\ |f(x) - A| < \epsilon,\ \ whenever\ \ 0 < |x - a| < \delta$$

This also fulfills the definition of the infinitesimals, so it actually means that:

$$f(x) - A = \alpha(x).$$

Now, let's suppose that $f(x) - A = \alpha(x)$, we need to show that : $\lim_{x \to a} f(x) = A\ \ ???$

Now, since $\alpha(x)$ is an infinitesimal, from its definition we have:

$$\forall\epsilon > 0, \exists\delta(\epsilon) > 0,\ \ \ such\ that |\alpha(x)| < \epsilon,\ \ whenever\ \ 0 < |x - a| < \delta.$$

Since, $f(x) - A = \alpha(x)$ we also get: $|\alpha(x) = f(x) - A| < \epsilon$.

So, $\lim_{x \to a} f(x) = A$. What we actually wanted to prove.

The following is the most important part of what I want to show. Now I will use **THEOREM (*)** as a definition of the limit of a function, and from there I will prove the limit laws.

Definition: A is said to be the limit of a function f(x) as x-->a, if and only if $f(x) - A = \alpha(x)$.

Theorem: If $\lim_{x \to a} f(x) = A,\ \ and\ \ \lim_{x \to a} g(x) = B$ then prove that:

1. $\lim_{x \to a}[f(x) + g(x)] = A + B = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. $\lim\limits_{x \to a}[f(x) - g(x)] = A - B = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$

3. $\lim\limits_{x \to a}[f(x) * g(x)] = A * B = \lim\limits_{x \to a} f(x) * \lim\limits_{x \to a} g(x)$

4. $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{A}{B} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \;\; where\ B = / = 0$

5. $\lim\limits_{x \to a}[c * f(x)] = c * A = c * \lim\limits_{x \to a} f(x)$

6. $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{A} = \sqrt[n]{\lim\limits_{x \to a} f(x)}$

Proof:

1.

Since $\lim\limits_{x \to a} f(x) = A$, it actually means that $f(x) - A = \alpha(x)$.
Also, since $\lim\limits_{x \to a} g(x) = B$, from the definition it means that $g(x) - B = \beta(x)$. Where both $\alpha(x) \quad and \quad \beta(x)$ are infinitesimals.

In order to prove this par of the theorem, we need to show that:

$[f(x) + g(x)] - [A + B] = \gamma(x) \quad ???$. Where $\gamma(x)$is also an infinitesimal. In other words we need to show that the difference $[f(x) + g(x)] - [A + B]$is actually an infinitesimal. Let that infinitesimal be any $\gamma(x)$.

So,

$[f(x)+g(x)]-[A+B] = [f(x)-A]+[g(x)-B] = \alpha(x)+\beta(x) = \gamma(x)$. We know from e previous theorem that the sum of two infinitesimals is again an infinitesimal.
So we proved this part of the theorem.