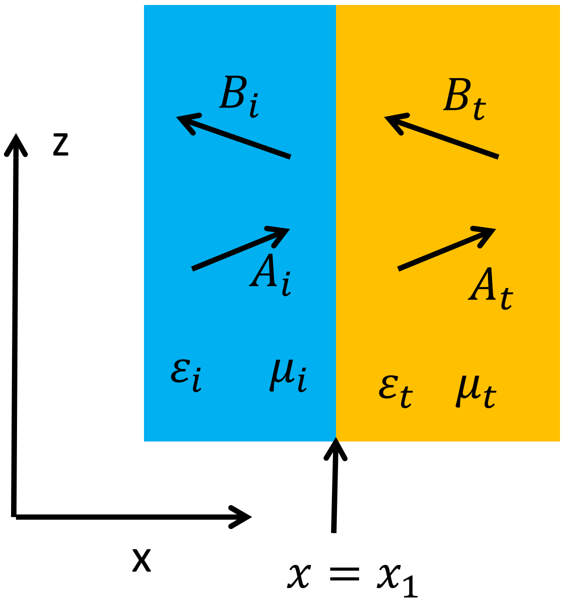
Case 2: TE, Total Internal Reflection



$$\vec{E}_t = e^{j(\omega t - \beta z)}\{A_t e^{-jk_{tx}(x-x_1)} + B_t e^{jk_{tx}(x-x_1)}\}\vec{e}_y$$

$$\beta^2 + k_{tx}^2 = \omega^2 \varepsilon_t \mu_t$$

With total internal reflection (TIR), we need: $\beta > \omega\sqrt{\varepsilon_t \mu_t}$

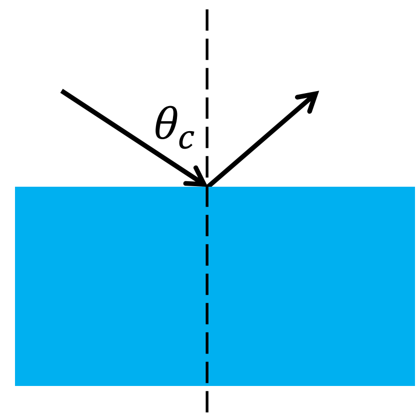
For dielectric material, this generally requires $\varepsilon_i > \varepsilon_t$. The incidence angle also must exceed the critical angleθ_c for TIR.

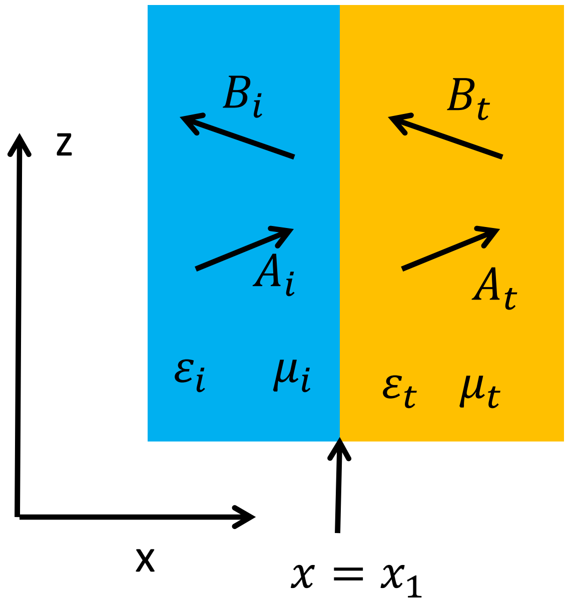
$$n_i = \sqrt{\varepsilon_i}$$

$$n_t = \sqrt{\varepsilon_t}$$

$$\mu_i = \mu_t = \mu_0$$

$$\sin\theta_c = \frac{n_t}{n_i}$$





Case 2: TE, Total Internal Reflection

$$\vec{E}_t = e^{j(\omega t-\beta z)}\{A_t e^{-jk_{tx}(x-x_1)} + B_t e^{jk_{tx}(x-x_1)}\}\vec{e}_y$$

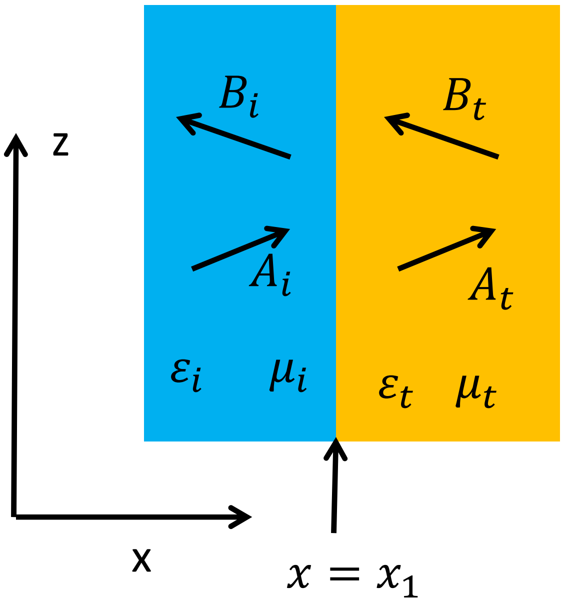
$$k_{tx} = \sqrt{\omega^2 \varepsilon_t \mu_t - \beta^2} = j\kappa_{tx}$$

$$\kappa_{tx} = \sqrt{\beta^2 - \omega^2 \varepsilon_t \mu_t} > 0$$

$$\vec{E}_t = e^{j(\omega t-\beta z)}\{A_t e^{\kappa_{tx}(x-x_1)} + B_t e^{-\kappa_{tx}(x-x_1)}\}\vec{e}_y$$

As a result, the boundary condition for the transmission layer is:

$$\begin{bmatrix} A_t \\ B_t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



Case 2: TE, TIR

$$D_i^{TE}\begin{bmatrix}A_i\\B_i\end{bmatrix} = D_t^{TE}\begin{bmatrix}A_t\\B_t\end{bmatrix}$$

$$D_n^{TE} = \begin{bmatrix}1 & 1\\ \dfrac{k_{nx}}{\mu_n} & -\dfrac{k_{nx}}{\mu_n}\end{bmatrix}$$

$$\begin{bmatrix}A_i\\B_i\end{bmatrix} = \left[D_i^{TE}\right]^{-1} D_t^{TE}\begin{bmatrix}A_t\\B_t\end{bmatrix}$$

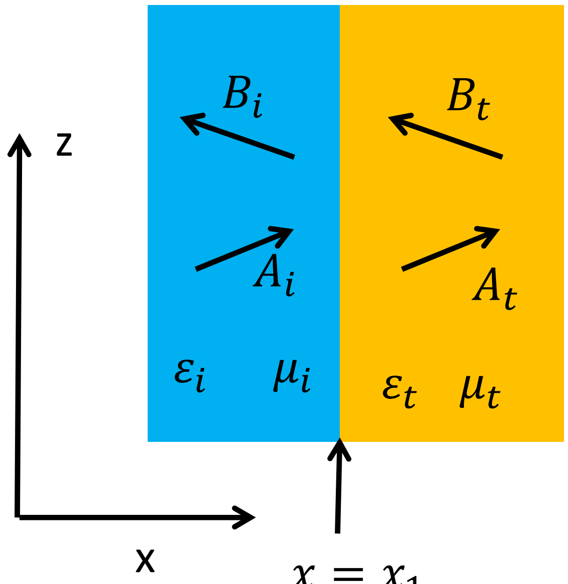
$$k_{tx} = j\kappa_{tx}$$

$$\begin{bmatrix}A_i\\B_i\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & \dfrac{\mu_i}{k_{ix}}\\ 1 & -\dfrac{\mu_i}{k_{ix}}\end{bmatrix}\begin{bmatrix}1 & 1\\ \dfrac{k_{tx}}{\mu_t} & -\dfrac{k_{tx}}{\mu_t}\end{bmatrix}\begin{bmatrix}A_t\\B_t\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1+\dfrac{\mu_i k_{tx}}{\mu_t k_{ix}} & 1-\dfrac{\mu_i k_{tx}}{\mu_t k_{ix}}\\ 1-\dfrac{\mu_i k_{tx}}{\mu_t k_{ix}} & 1+\dfrac{\mu_i k_{tx}}{\mu_t k_{ix}}\end{bmatrix}\begin{bmatrix}A_t\\B_t\end{bmatrix}$$

$$\begin{bmatrix}A_t\\B_t\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

$$\begin{bmatrix}A_i\\B_i\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1-\dfrac{\mu_i k_{tx}}{\mu_t k_{ix}}\\ 1+\dfrac{\mu_i k_{tx}}{\mu_t k_{ix}}\end{bmatrix}$$

Case 2: TE, TIR



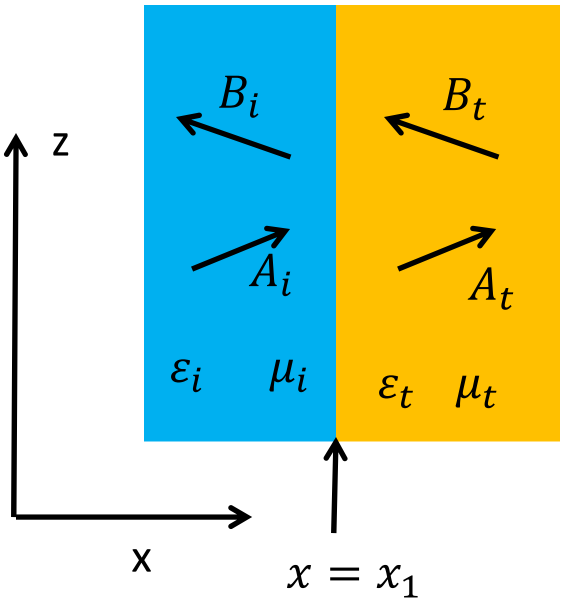
$$\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} A_i \\ B_i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 - \dfrac{\mu_i k_{tx}}{\mu_t k_{ix}} \\ 1 + \dfrac{\mu_i k_{tx}}{\mu_t k_{ix}} \end{bmatrix}$$

$$k_{ix} = \sqrt{\omega^2 \varepsilon_i \mu_i - \beta^2} > 0 \qquad k_{tx} = \sqrt{\omega^2 \varepsilon_t \mu_t - \beta^2} = j\kappa_{tx}$$

$$r = \frac{B_i}{A_i} = \frac{1 + \dfrac{\mu_i k_{tx}}{\mu_t k_{ix}}}{1 - \dfrac{\mu_i k_{tx}}{\mu_t k_{ix}}} = \frac{1 + j\dfrac{\mu_i \kappa_{tx}}{\mu_t k_{ix}}}{1 - j\dfrac{\mu_i \kappa_{tx}}{\mu_t k_{ix}}} = e^{j\phi} \qquad R = \left|\frac{B_i}{A_i}\right|^2 = 1$$

The reflected wave experiences a phase shift but no decrease in amplitude.

Case 2: TE, TIR



$$\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \vec{E}_t = e^{j(\omega t - \beta z)} B_t e^{-\kappa_{tx}(x - x_1)} \vec{e}_y$$

We can now calculate the Poynting Vector

$$\vec{H}_t = \frac{j}{\omega \mu_t} \nabla \times \vec{E}_t = \frac{j}{\omega \mu_t} \left[\frac{\partial E_{ty}}{\partial x} \vec{e}_z - \frac{\partial E_{ty}}{\partial z} \vec{e}_x \right]$$

$$H_{tx} = -\frac{j}{\omega \mu_t} \frac{\partial E_{ty}}{\partial z}$$

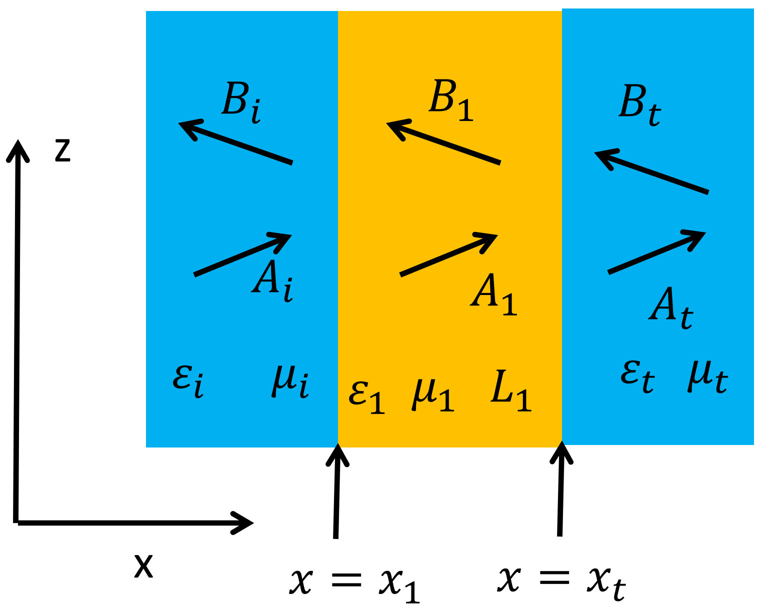
$$H_{tz} = \frac{j}{\omega \mu_t} \frac{\partial E_{ty}}{\partial x}$$

$$H_{tx} = -\frac{\beta}{\omega \mu_t} e^{j(\omega t - \beta z)} B_t e^{-\kappa_{tx}(x - x_1)}$$

$$H_{tz} = -\frac{j\kappa_{tx}}{\omega \mu_t} e^{j(\omega t - \beta z)} B_t e^{-\kappa_{tx}(x - x_1)}$$

$$\vec{S} = \frac{\beta}{2\omega \mu_t} |B_t|^2 e^{-2\kappa_{tx}(x - x_1)} \vec{e}_z$$

Case 3: TE, Dielectric Waveguide



$$D_n^{TE} = \begin{bmatrix} 1 & 1 \\ \frac{k_{nx}}{\mu_n} & -\frac{k_{nx}}{\mu_n} \end{bmatrix}$$

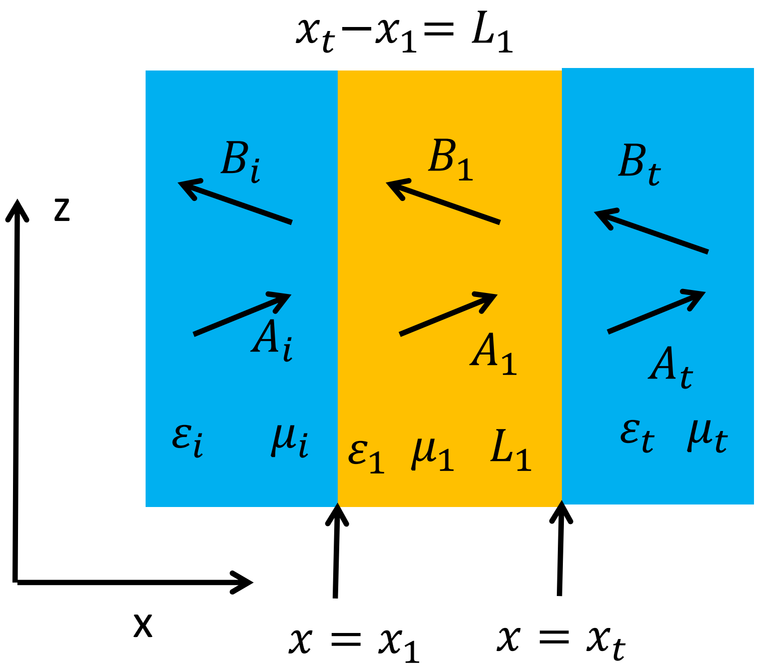
$$P_n^{TE} = \begin{bmatrix} e^{-jk_{nx}L_n} & 0 \\ 0 & e^{jk_{nx}L_n} \end{bmatrix}$$

$$x_t - x_1 = L_1$$

$$\begin{bmatrix} A_t \\ B_t \end{bmatrix} = T_{TE} \begin{bmatrix} A_i \\ B_i \end{bmatrix}$$

$$T_{TE} = [(D_t^{TE})^{-1} D_1^{TE} P_1^{TE}][(D_1^{TE})^{-1} D_i^{TE}]$$

Case 3: TE, Dielectric Waveguide



$$\begin{bmatrix} A_t \\ B_t \end{bmatrix} = \begin{bmatrix} (T_{TE})_{11} & (T_{TE})_{12} \\ (T_{TE})_{21} & (T_{TE})_{22} \end{bmatrix} \begin{bmatrix} A_i \\ B_i \end{bmatrix}$$

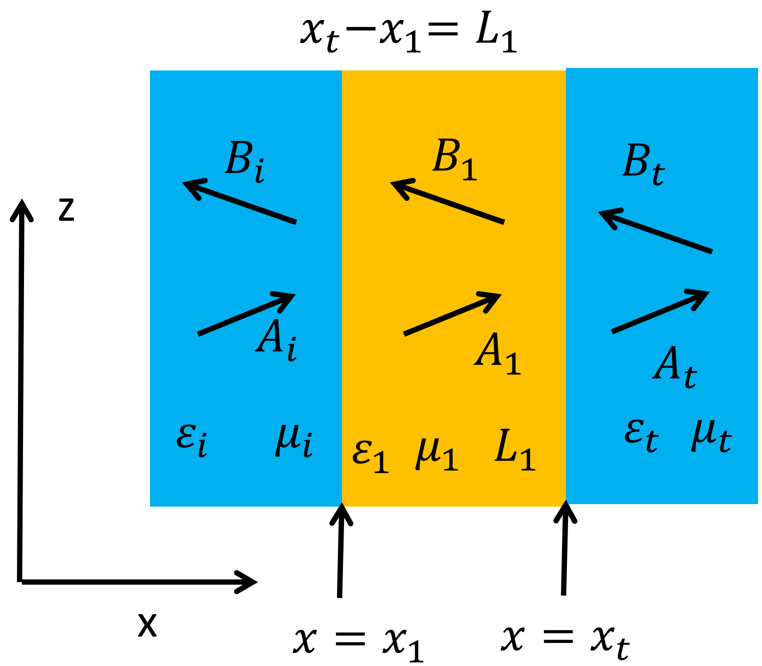
$$\vec{E}_t = e^{j(\omega t - \beta z)} \{ A_t e^{-jk_{tx}(x - x_t)} + B_t e^{jk_{tx}(x - x_t)} \} \vec{e}_y$$

$$k_{tx} = \sqrt{\omega^2 \varepsilon_t \mu_t - \beta^2} = j\kappa_{tx}$$

$$\vec{E}_1 = e^{j(\omega t - \beta z)} \{ A_1 e^{-jk_{1x}(x - x_1)} + B_1 e^{jk_{1x}(x - x_1)} \} \vec{e}_y \qquad k_{1x} = \sqrt{\omega^2 \varepsilon_1 \mu_1 - \beta^2} > 0$$

$$\vec{E}_i = e^{j(\omega t - \beta z)} \{ A_i e^{-jk_{ix}(x - x_1)} + B_i e^{jk_{ix}(x - x_1)} \} \vec{e}_y \qquad k_{ix} = \sqrt{\omega^2 \varepsilon_i \mu_i - \beta^2} = j\kappa_{ix}$$

Case 3: Waveguide Boundary Conditions



$$\begin{bmatrix} A_t \\ B_t \end{bmatrix} = \begin{bmatrix} (T_{TE})_{11} & (T_{TE})_{12} \\ (T_{TE})_{21} & (T_{TE})_{22} \end{bmatrix} \begin{bmatrix} A_i \\ B_i \end{bmatrix}$$

$$\vec{E}_t = e^{j(\omega t - \beta z)} \{ A_t e^{\kappa_{tx}(x - x_t)} + B_t e^{-\kappa_{tx}(x - x_t)} \} \vec{e}_y$$

Boundary Condition: $A_t = 0$

$$\vec{E}_i = e^{j(\omega t - \beta z)} \{ A_i e^{\kappa_{ix}(x - x_1)} + B_i e^{-\kappa_{ix}(x - x_1)} \} \vec{e}_y$$

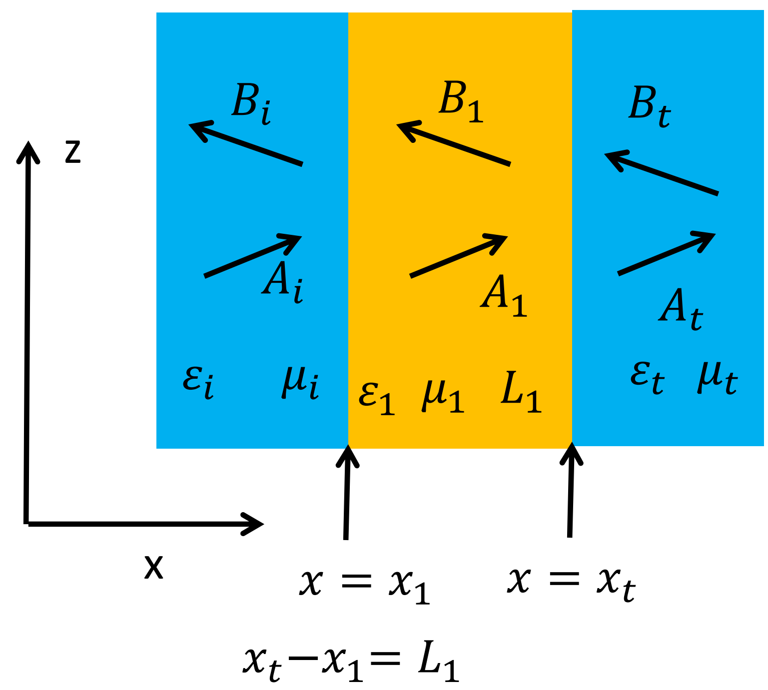
Boundary Condition: $B_i = 0$

$$\begin{bmatrix} 0 \\ B_t \end{bmatrix} = \begin{bmatrix} (T_{TE})_{11} & (T_{TE})_{12} \\ (T_{TE})_{21} & (T_{TE})_{22} \end{bmatrix} \begin{bmatrix} A_i \\ 0 \end{bmatrix}$$

The equation that determines propagation constant β:

$$(T_{TE})_{11} = 0$$

Case 3: How to Find Waveguide Modes



$$(T_{TE})_{11} = 0$$

- The equation (highlighted in yellow) is the one that determines waveguide modes.
- If frequency, structural parameters and material properties are given, the transfer matrix will only depend on the value of β.
- Generally speaking, at any given frequency, there are only a discrete number of waveguide modes, each associated with a specific propagation constant β. The number of such modes, however, can be very large depending on waveguide parameters.

$$T_{TE} = [(D_t^{TE})^{-1} D_1^{TE} P_1^{TE}][(D_1^{TE})^{-1} D_i^{TE}]$$

$$k_{tx} = \sqrt{\omega^2 \varepsilon_t \mu_t - \beta^2} = j\kappa_{tx}$$

$$D_n^{TE} = \begin{bmatrix} 1 & 1 \\ \dfrac{k_{nx}}{\mu_n} & -\dfrac{k_{nx}}{\mu_n} \end{bmatrix} \qquad P_n^{TE} = \begin{bmatrix} e^{-jk_{nx}L_n} & 0 \\ 0 & e^{jk_{nx}L_n} \end{bmatrix}$$

$$k_{1x} = \sqrt{\omega^2 \varepsilon_1 \mu_1 - \beta^2} > 0$$

$$k_{ix} = \sqrt{\omega^2 \varepsilon_i \mu_i - \beta^2} = j\kappa_{ix}$$

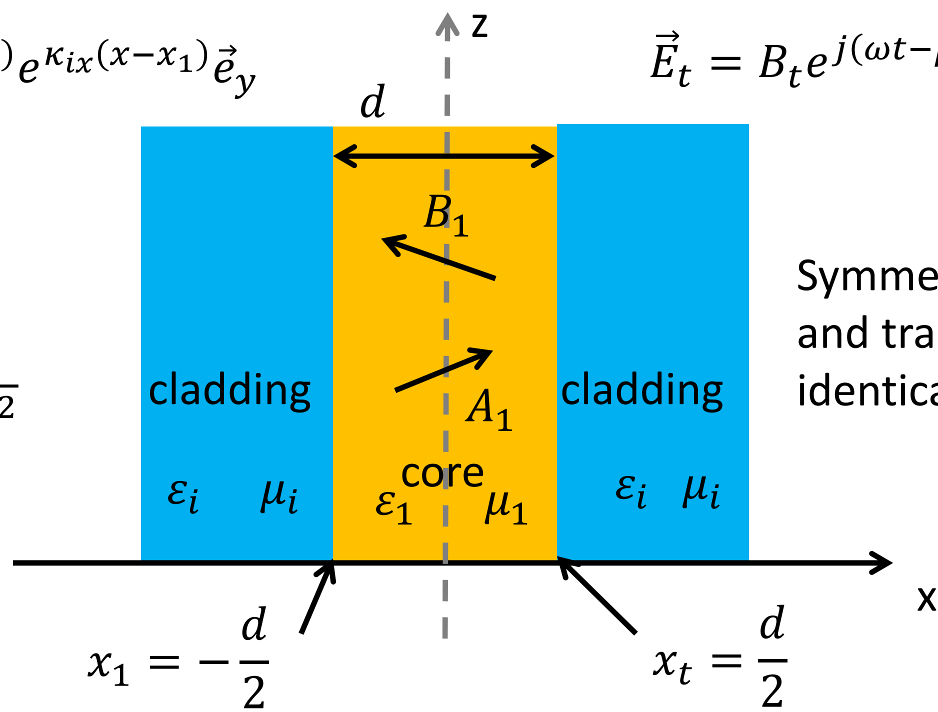
Case 3: TE Modes of a Symmetric Slab

$$\vec{E}_i = A_i e^{j(\omega t - \beta z)} e^{\kappa_{ix}(x - x_1)} \vec{e}_y$$

$$\vec{E}_t = B_t e^{j(\omega t - \beta z)} e^{-\kappa_{ix}(x - x_t)} \vec{e}_y$$

$$\kappa_{ix} = \sqrt{\beta^2 - \omega^2 \varepsilon_i \mu_i}$$

$$k_{1x} = \sqrt{\omega^2 \varepsilon_1 \mu_1 - \beta^2}$$



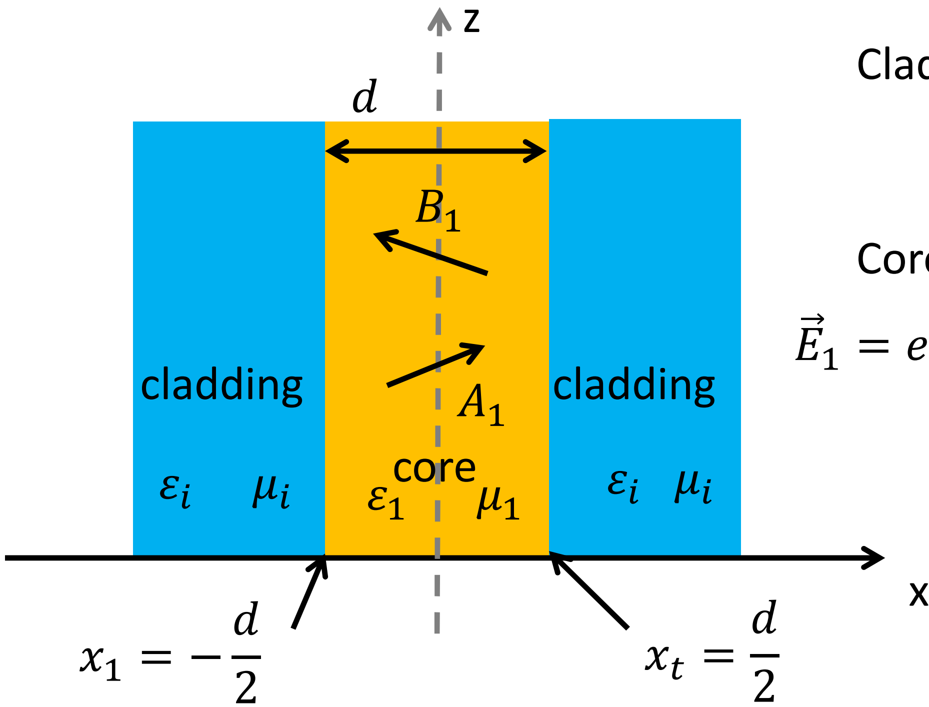
Symmetric Slab: Incidence and transmission layers are identical.

$$\vec{E}_1 = e^{j(\omega t - \beta z)} \{ A_1 e^{-jk_{1x}(x - x_1)} + B_1 e^{jk_{1x}(x - x_1)} \} \vec{e}_y$$

Due to the symmetry of the waveguide geometry, the electric field in the core region can be expressed as either a sin or a cos function.

$$\vec{E}_1 \propto e^{j(\omega t - \beta z)} \cos k_{1x} x \, \vec{e}_y \qquad \qquad \vec{E}_1 \propto e^{j(\omega t - \beta z)} \sin k_{1x} x \, \vec{e}_y$$

Symmetric TE Mode



Cladding: $\vec{E}_i = A_i e^{j(\omega t - \beta z)} e^{\kappa_{ix}(x - x_1)} \vec{e}_y$

$$\boxed{B_i = 0}$$

Core:

$$\vec{E}_1 = e^{j(\omega t - \beta z)} \{ A_1 e^{-jk_{1x}(x - x_1)} + B_1 e^{jk_{1x}(x - x_1)} \} \vec{e}_y$$

In order to achieve the symmetric form:

$$\vec{E}_1 \propto e^{j(\omega t - \beta z)} \cos k_{1x} x \, \vec{e}_y$$

We have:

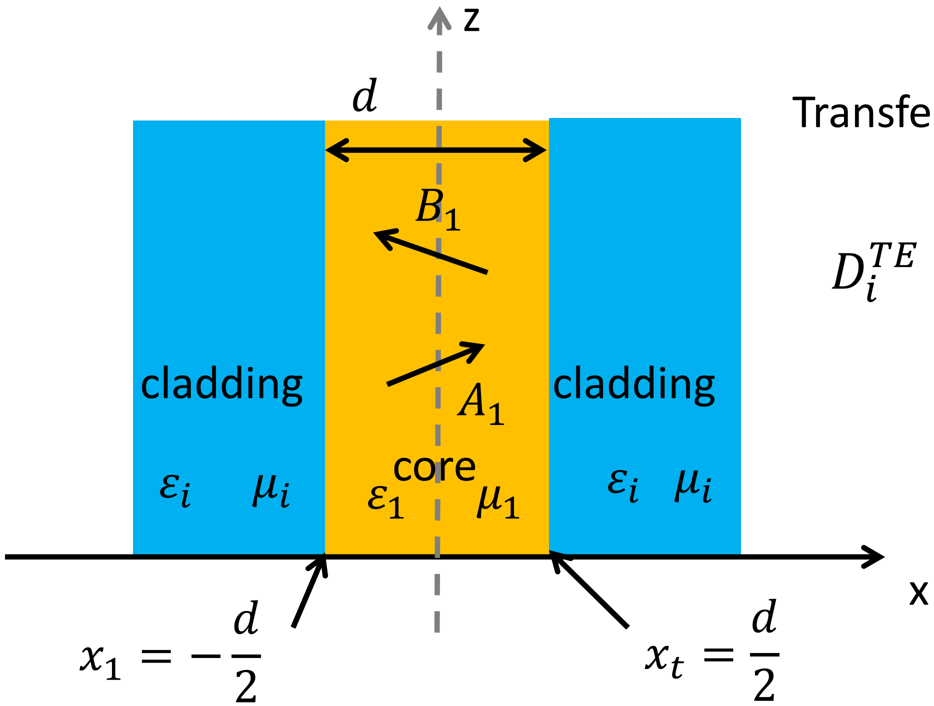
$$\boxed{A_1 = e^{jk_{1x}d/2}}$$

$$\boxed{B_1 = e^{-jk_{1x}d/2}}$$

$$\kappa_{ix} = \sqrt{\beta^2 - \omega^2 \varepsilon_i \mu_i}$$

$$k_{1x} = \sqrt{\omega^2 \varepsilon_1 \mu_1 - \beta^2}$$

Symmetric TE Mode



Transfer matrix between the cladding and core layer:

$$D_i^{TE}\begin{bmatrix}A_i\\B_i\end{bmatrix} = D_1^{TE}\begin{bmatrix}A_1\\B_1\end{bmatrix}$$

$$D_n^{TE} = \begin{bmatrix}1 & 1\\ \frac{k_{nx}}{\mu_n} & -\frac{k_{nx}}{\mu_n}\end{bmatrix}$$

$$\begin{bmatrix}A_i\\B_i\end{bmatrix} = \left[D_i^{TE}\right]^{-1} D_1^{TE}\begin{bmatrix}A_1\\B_1\end{bmatrix}$$

$$\begin{bmatrix}A_i\\B_i\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & \frac{\mu_i}{j\kappa_{ix}}\\ 1 & -\frac{\mu_i}{j\kappa_{ix}}\end{bmatrix}\begin{bmatrix}1 & 1\\ \frac{k_{1x}}{\mu_1} & -\frac{k_{1x}}{\mu_1}\end{bmatrix}\begin{bmatrix}A_1\\B_1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1+\frac{\mu_i k_{1x}}{j\mu_1\kappa_{ix}} & 1-\frac{\mu_i k_{1x}}{j\mu_1\kappa_{ix}}\\ 1-\frac{\mu_i k_{1x}}{j\mu_1\kappa_{ix}} & 1+\frac{\mu_i k_{1x}}{j\mu_1\kappa_{ix}}\end{bmatrix}\begin{bmatrix}A_1\\B_1\end{bmatrix}$$

$$\kappa_{ix} = \sqrt{\beta^2 - \omega^2\varepsilon_i\mu_i} \qquad k_{1x} = \sqrt{\omega^2\varepsilon_1\mu_1 - \beta^2}$$

Symmetric TE Mode

$$\begin{bmatrix} A_i \\ B_i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+\frac{\mu_i k_{1x}}{j\mu_1\kappa_{ix}} & 1-\frac{\mu_i k_{1x}}{j\mu_1\kappa_{ix}} \\ 1-\frac{\mu_i k_{1x}}{j\mu_1\kappa_{ix}} & 1+\frac{\mu_i k_{1x}}{j\mu_1\kappa_{ix}} \end{bmatrix}\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}$$

$$B_i = 0$$

$$A_1 = e^{jk_{1x}d/2}$$

$$B_1 = e^{-jk_{1x}d/2}$$

$$\begin{bmatrix} A_i \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+\frac{\mu_i k_{1x}}{j\mu_1\kappa_{ix}} & 1-\frac{\mu_i k_{1x}}{j\mu_1\kappa_{ix}} \\ 1-\frac{\mu_i k_{1x}}{j\mu_1\kappa_{ix}} & 1+\frac{\mu_i k_{1x}}{j\mu_1\kappa_{ix}} \end{bmatrix}\begin{bmatrix} e^{jk_{1x}d/2} \\ e^{-jk_{1x}d/2} \end{bmatrix}$$

$$\kappa_{ix} = \sqrt{\beta^2 - \omega^2\varepsilon_i\mu_i}$$

$$k_{1x} = \sqrt{\omega^2\varepsilon_1\mu_1 - \beta^2}$$

$$\left(1-\frac{\mu_i k_{1x}}{j\mu_1\kappa_{ix}}\right)e^{jk_{1x}d/2} + \left(1+\frac{\mu_i k_{1x}}{j\mu_1\kappa_{ix}}\right)e^{-jk_{1x}d/2} = 0$$

$$e^{jk_{1x}d} = -\frac{1+\frac{\mu_i k_{1x}}{j\mu_1\kappa_{ix}}}{1-\frac{\mu_i k_{1x}}{j\mu_1\kappa_{ix}}} = \frac{\frac{\mu_i k_{1x}}{\mu_1\kappa_{ix}}+j}{\frac{\mu_i k_{1x}}{\mu_1\kappa_{ix}}-j}$$

Symmetric TE Mode

$$e^{jk_{1x}d} = -\frac{1+\dfrac{\mu_i k_{1x}}{j\mu_1\kappa_{ix}}}{1-\dfrac{\mu_i k_{1x}}{j\mu_1\kappa_{ix}}} = \frac{\dfrac{\mu_i k_{1x}}{\mu_1\kappa_{ix}}+j}{\dfrac{\mu_i k_{1x}}{\mu_1\kappa_{ix}}-j}$$

Set: $$\frac{\cos\dfrac{\phi}{2}}{\sin\dfrac{\phi}{2}} = \frac{\dfrac{\mu_i k_{1x}}{\mu_1\kappa_{ix}}}{1}$$

$$e^{jk_{1x}d} = \frac{\cos\dfrac{\phi}{2}+j\sin\dfrac{\phi}{2}}{\cos\dfrac{\phi}{2}-j\sin\dfrac{\phi}{2}} = e^{j\phi}$$

$$\phi = k_{1x}d$$

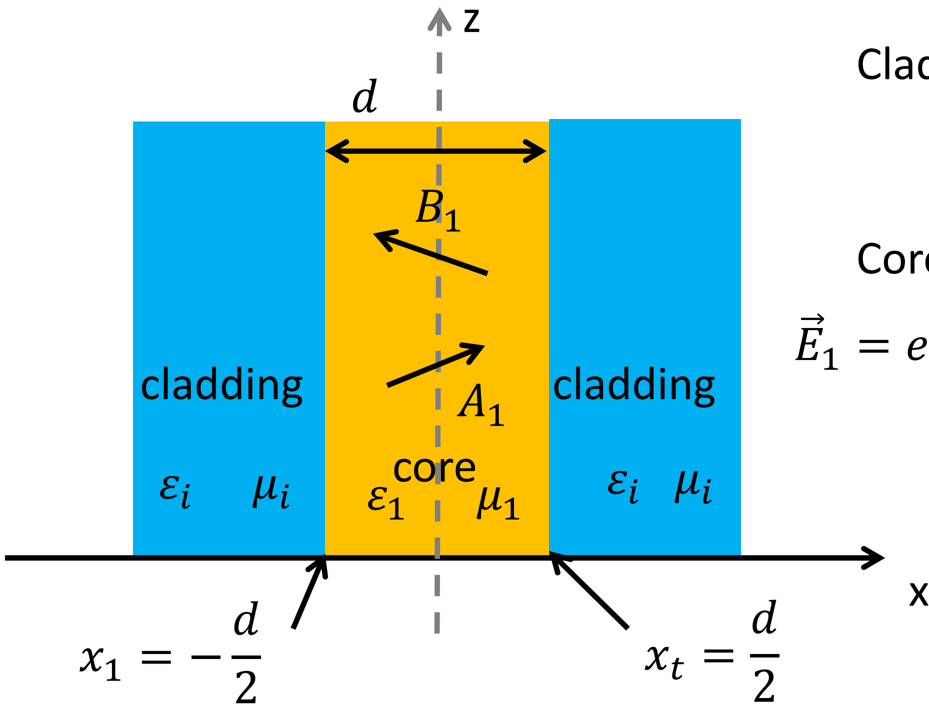
$$\tan\frac{k_{1x}d}{2} = \frac{\mu_1\kappa_{ix}}{\mu_i k_{1x}}$$

$$\mu_i\frac{k_{1x}d}{2}\tan\frac{k_{1x}d}{2} = \mu_1\frac{\kappa_{ix}d}{2}$$

$$k_{1x} = \sqrt{\omega^2\varepsilon_1\mu_1-\beta^2}$$

$$\kappa_{ix} = \sqrt{\beta^2-\omega^2\varepsilon_i\mu_i}$$

Anti-Symmetric TE Mode



Cladding: $\vec{E}_i = A_i e^{j(\omega t - \beta z)} e^{\kappa_{ix}(x - x_1)} \vec{e}_y$

$$B_i = 0$$

Core:

$$\vec{E}_1 = e^{j(\omega t - \beta z)} \{A_1 e^{-jk_{1x}(x - x_1)} + B_1 e^{jk_{1x}(x - x_1)}\} \vec{e}_y$$

In order to achieve the anti-symmetric form:

$$\vec{E}_1 \propto e^{j(\omega t - \beta z)} \sin k_{1x} x \, \vec{e}_y$$

We have:

$$A_1 = e^{jk_{1x} d/2}$$

$$B_1 = -e^{-jk_{1x} d/2}$$

$$\kappa_{ix} = \sqrt{\beta^2 - \omega^2 \varepsilon_i \mu_i}$$

$$k_{1x} = \sqrt{\omega^2 \varepsilon_1 \mu_1 - \beta^2}$$

Anti-Symmetric TE Mode

$$\begin{bmatrix} A_i \\ B_i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+\dfrac{\mu_i k_{1x}}{j\mu_1 \kappa_{ix}} & 1-\dfrac{\mu_i k_{1x}}{j\mu_1 \kappa_{ix}} \\ 1-\dfrac{\mu_i k_{1x}}{j\mu_1 \kappa_{ix}} & 1+\dfrac{\mu_i k_{1x}}{j\mu_1 \kappa_{ix}} \end{bmatrix}\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}$$

$$B_i = 0$$

$$A_1 = e^{jk_{1x}d/2}$$

$$B_1 = -e^{-jk_{1x}d/2}$$

$$\begin{bmatrix} A_i \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+\dfrac{\mu_i k_{1x}}{j\mu_1 \kappa_{ix}} & 1-\dfrac{\mu_i k_{1x}}{j\mu_1 \kappa_{ix}} \\ 1-\dfrac{\mu_i k_{1x}}{j\mu_1 \kappa_{ix}} & 1+\dfrac{\mu_i k_{1x}}{j\mu_1 \kappa_{ix}} \end{bmatrix}\begin{bmatrix} e^{jk_{1x}d/2} \\ -e^{-jk_{1x}d/2} \end{bmatrix}$$

$$\kappa_{ix} = \sqrt{\beta^2 - \omega^2 \varepsilon_i \mu_i}$$

$$k_{1x} = \sqrt{\omega^2 \varepsilon_1 \mu_1 - \beta^2}$$

$$\left(1-\frac{\mu_i k_{1x}}{j\mu_1 \kappa_{ix}}\right)e^{jk_{1x}d/2} - \left(1+\frac{\mu_i k_{1x}}{j\mu_1 \kappa_{ix}}\right)e^{-jk_{1x}d/2} = 0$$

$$e^{jk_{1x}d} = \frac{1+\dfrac{\mu_i k_{1x}}{j\mu_1 \kappa_{ix}}}{1-\dfrac{\mu_i k_{1x}}{j\mu_1 \kappa_{ix}}} = \frac{\dfrac{\mu_1 \kappa_{ix}}{\mu_i k_{1x}} - j}{\dfrac{\mu_1 \kappa_{ix}}{\mu_i k_{1x}} + j}$$

Anti-Symmetric TE Mode

$$e^{jk_{1x}d} = \frac{\dfrac{\mu_1 \kappa_{ix}}{\mu_i k_{1x}} - j}{\dfrac{\mu_1 \kappa_{ix}}{\mu_i k_{1x}} + j}$$

Set: $$\frac{\cos\dfrac{\phi}{2}}{\sin\dfrac{\phi}{2}} = \frac{\mu_1 \kappa_{ix}}{\mu_i k_{1x}}$$

$$e^{jk_{1x}d} = \frac{\cos\dfrac{\phi}{2} - j\sin\dfrac{\phi}{2}}{\cos\dfrac{\phi}{2} + j\sin\dfrac{\phi}{2}} = e^{-j\phi}$$

$$\phi = -\,k_{1x}d$$

$$-\cot\frac{k_{1x}d}{2} = \frac{\mu_1 \kappa_{ix}}{\mu_i k_{1x}}$$

$$-\mu_i \frac{k_{1x}d}{2} \cot\frac{k_{1x}d}{2} = \mu_1 \frac{\kappa_{ix}d}{2}$$

$$k_{1x} = \sqrt{\omega^2 \varepsilon_1 \mu_1 - \beta^2}$$

$$\kappa_{ix} = \sqrt{\beta^2 - \omega^2 \varepsilon_i \mu_i}$$

Summary of TE Modes

Symmetric:

$$\mu_i \frac{k_{1x}d}{2} \tan \frac{k_{1x}d}{2} = \mu_1 \frac{\kappa_{ix}d}{2}$$

Anti-Symmetric:

$$-\mu_i \frac{k_{1x}d}{2} \cot \frac{k_{1x}d}{2} = \mu_1 \frac{\kappa_{ix}d}{2}$$

d: waveguide thickness.
β: propagation constant.

$$k_{1x} = \sqrt{\omega^2 \varepsilon_1 \mu_1 - \beta^2}$$

$$\kappa_{ix} = \sqrt{\beta^2 - \omega^2 \varepsilon_i \mu_i}$$

$$\omega\sqrt{\varepsilon_i \mu_i} < \beta < \omega\sqrt{\varepsilon_1 \mu_1}$$

$$\left(\frac{k_{1x}d}{2}\right)^2 + \left(\frac{\kappa_{1x}d}{2}\right)^2 = \left(\frac{\omega d}{2}\right)^2 (\varepsilon_1 \mu_1 - \varepsilon_i \mu_i)$$

How to Solve TE Modes

Symmetric: $\mu_i x \tan x = \mu_1 y$

Anti-Symmetric: $-\mu_i x \cot x = \mu_1 y$

$$x = \frac{k_{1x} d}{2}$$

$$y = \frac{\kappa_{ix} d}{2}$$

$$x^2 + y^2 = u^2$$

$$u = \frac{\omega d}{2}\sqrt{\varepsilon_1 \mu_1 - \varepsilon_i \mu_i}$$

For most dielectric materials, we have $\mu_1 = \mu_i = \mu_0$

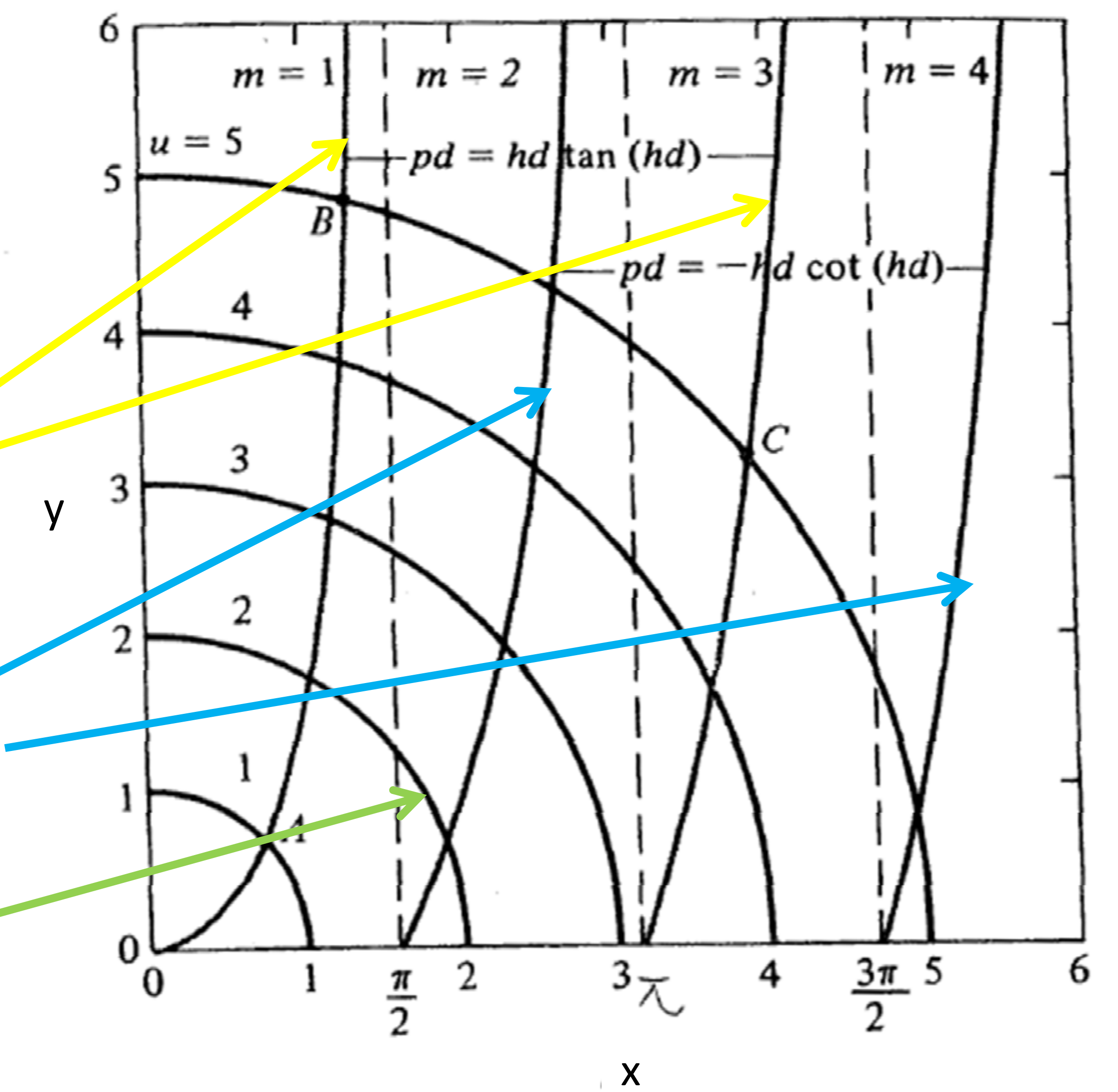
Numerical Solution for TE Modes

$\mu_1 = \mu_i = \mu_0$

$x\tan x = y$

$-x\cot x = y$

$x^2 + y^2 = u^2$



How to determine the number of TE Modes

n2: cladding index $n_2 = 1.0$

n1: core index $n_1 = 1.5$

d: waveguide thickness $d = 1 \mu m$

Wavelength (in air): $\lambda = 1.55 \ \mu m$

Non-magnetic materials: $\mu_1 = \mu_i = \mu_0$

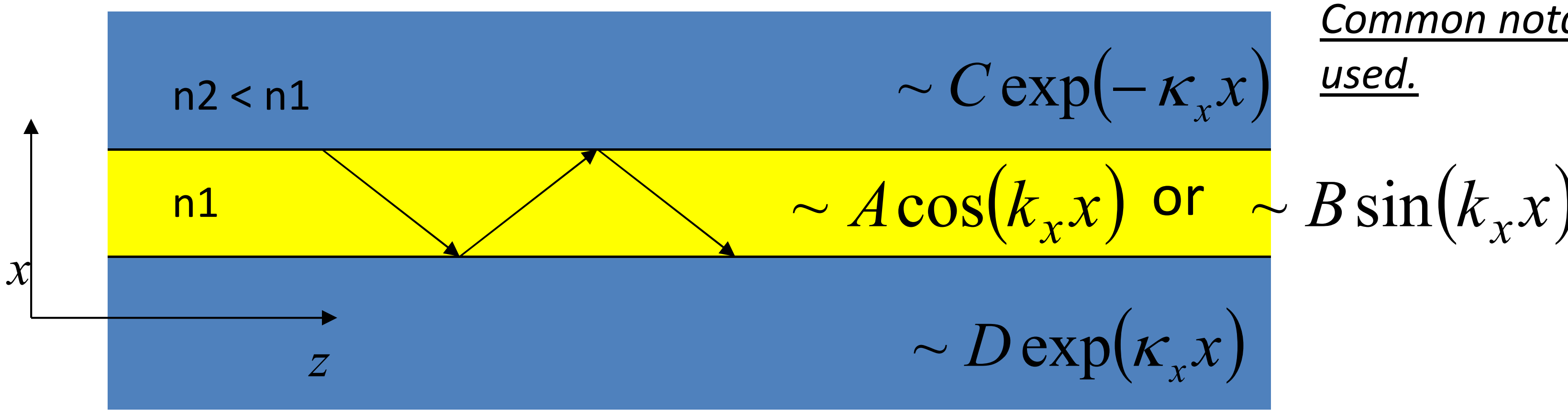
$$\left(\frac{k_{1x}d}{2}\right)^2 + \left(\frac{\kappa_{1x}d}{2}\right)^2 = \left(\frac{\omega d}{2}\right)^2 (\varepsilon_1\mu_1 - \varepsilon_i\mu_i)$$

$$u = \frac{\omega d}{2}\sqrt{\varepsilon_1\mu_1 - \varepsilon_i\mu_i}$$

$$u = \sqrt{n_1^2 - n_2^2}\,\frac{2\pi}{\lambda}\frac{d}{2} = 2.2661$$

TE Modes	**Cutoff value for u**
TE_1 (symmetric)	$u = 0$
TE_2 (anti-symmetric)	$u = \frac{\pi}{2} = 1.5708$
TE_3 (symmetric)	$u = \pi = 3.1416$
TE_4 (anti-symmetric)	$u = \frac{3}{2}\pi = 4.7124$
TE_5 (symmetric)	$u = 2\pi = 6.2832$

A Few Comments on <u>Dielectric</u> Waveguide Modes



<u>Common notation used.</u>

$$n_2\frac{\omega}{c} < \beta < n_1\frac{\omega}{c}$$

$$\kappa_x = \sqrt{\beta^2 - n_2^2\frac{\omega^2}{c^2}}$$

$$k_x = \sqrt{n_1^2\frac{\omega^2}{c^2} - \beta^2}$$

A few important comments:

1. The value of propagation constant β can only exist between two limits;
2. There exist only a finite number of waveguide modes.
3. The propagation constant β is one of the most important quantity for any waveguide.
4. The physical meaning of κ_x and k_x. In particular, from their value we can estimate the "incidence angle" and "penetration into the cladding layer".
5. Note that there a few unknown coefficients in the field expression, and we need to solve certain equations to obtain the propagation constant β and the mode profile.

Qualitative Analysis of Waveguide Modes

Consider the fundamental TE mode.
If core + cladding index are fixed, at any given wavelength λ, d becomes larger and larger.

1. Would β increase or decrease?

- β would increase but cannot exceed the limit imposed by: $\beta < \omega\sqrt{\varepsilon_1\mu_1}$

2. Would k_x & κ_x increase or decrease?

- $k_{1x}d$ would increase but cannot exceed the limit of: $\frac{k_{1x}d}{2} < \frac{\pi}{2}$
- For a sufficiently large d, k_{1x} would actually decrease.
- κ_{ix} would increase but cannot exceed the limit of: $\kappa_{ix} < \sqrt{\omega^2\varepsilon_1\mu_1 - \omega^2\varepsilon_i\mu_i}$

3. How would the spatial profile of the waveguide mode change?

The waveguide E field will increasingly look like it will be “pinched off” at the core/cladding boundary. In other words, the E field will increasingly look like a sine function that becomes zero at the core/cladding boundary.