Inverse of a tensor

By George Keeling, January 2021

Notation

All the matrices here are square $n \times n$ matrices. Tensors and the Levi-Civita symbol ϵ are indexed from $1-n$ so $\epsilon^{1234}=+1, \epsilon^{1243}=-1, \epsilon^{1434}=0$ etc. The indices on ϵ can be up or down to suit the equation.$|A|$ is the determinant of A and may be negative. $\bar{A}^{ij}$ is the inverse of A_{ij} so $\bar{A}^{ij}A_{jk}=\delta^i_k$ and $\bar{A}^{ij} \neq A^{ij}$ unless A is the metric. The determinant of the metric is often written just g instead of $|g|$.

We will prove that

$$\boxed{|A|\bar{A}^{\alpha\mu_\alpha} = \epsilon^{\mu_1\mu_2\mu_3\cdots\mu_n}\prod_{\beta\neq\alpha} A_{\mu_\beta\beta} \qquad (1)}$$

When applied to the metric it is

$$gg^{\alpha\mu_\alpha} = \epsilon^{\mu_1\mu_2\mu_3\cdots\mu_n}\prod_{\beta\neq\alpha} g_{\mu_\beta\beta} \tag{2}$$

which in four dimensions is

$$gg^{\alpha\mu_\alpha} = \epsilon^{\mu_1\mu_2\mu_3\mu_4}\prod_{\beta\neq\alpha} g_{\mu_\beta\beta} \tag{3}$$

Which is not the same as

$$gg^{\mu_\nu\nu} = (-1)^{\mu_\nu}\epsilon^{\mu_0\mu_1\mu_2\mu_3}\prod_{\lambda\neq\nu} g_{\mu_\lambda\lambda} \tag{4}$$

(indexing is over 0,1,2,3 in this formula only). Everything is fine in (4) apart from the pesky $(-1)^{\mu_\nu}$.

Proof of (1)

We agree that the determinant of a fully covariant tensor is

$$|A| = \epsilon^{\mu_1\mu_2\cdots\mu_n}A_{\mu_1 1}A_{\mu_2 2}\cdots A_{\mu_n n} \tag{5}$$

We want to extract the $A_{\mu_i i}$ term from that. We have a notation where the product sign is over terms $1-n$ with gaps and at (7) where the it occurs in the index of ϵ it only means a list of the operand. So if $i=3$, $\epsilon^{\mu_i\prod_{k\neq i}\mu_k} = \epsilon^{\mu_3\mu_1\mu_2\mu_4\cdots\mu_n}$. We get

$$\begin{aligned}
|A| &= A_{\mu_i i}\epsilon^{\mu_1\mu_2\cdots\mu_n}A_{\mu_1 1}A_{\mu_2 2}\cdots A_{\mu_{(i-1)}(i-1)}A_{\mu_{(i+1)}(i+1)}\cdots A_{\mu_n n} && (6)\\
&= A_{\mu_i i}(-1)^{i+1}\epsilon^{\mu_i\prod_{k\neq i}\mu_k}\prod_{j\neq i}A_{\mu_j j} && (7)\\
&= A_{1i}(-1)^{i+1}\epsilon^{1\prod_{\mu_k\neq 1}\mu_k}\prod_{j\neq i}A_{\mu_j j} && (8)\\
&+A_{2i}(-1)^{i+1}\epsilon^{2\prod_{\mu_k\neq 2}\mu_k}\prod_{j\neq i}A_{\mu_j j}\\
&\vdots\\
&+A_{ni}(-1)^{i+1}\epsilon^{n\prod_{\mu_k\neq n}\mu_k}\prod_{j\neq i}A_{\mu_j j} \quad \text{(no sum over } n\text{)}
\end{aligned}$$

Now consider the matrix representation of A and its cofactor matrix C

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}, \mathrm{adj}(A) = \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & c_{22} & \cdots & c_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1n} & c_{2n} & \cdots & c_{nn} \end{bmatrix} \tag{9}$$

(For perfection, the indices on c_{ij} should be upstairs. We will remedy that). We know that the determinant could be calculated going down the ith column of the matrix and it would be

$$|A| = a_{1i}c_{1i} + a_{2i}c_{2i} + \cdots + a_{ni}c_{ni} \tag{10}$$

Comparing that with (8) we see that

$$c_{1i} = (-1)^{i+1}\epsilon^{1\prod_{\mu_k \neq 1}\mu_k} \prod_{j \neq i} A_{\mu_j j} \tag{11}$$

In general

$$c_{li} = (-1)^{i+1}\epsilon^{l\prod_{\mu_k \neq l}\mu_k} \prod_{j \neq i} A_{\mu_j j} \tag{12}$$

Now set $l = \mu_i$ and that is

$$c_{\mu_i i} = (-1)^{i+1}\epsilon^{\mu_i\prod_{\mu_k \neq \mu_i}\mu_k} \prod_{j \neq i} A_{\mu_j j} \tag{13}$$

and push forward the μ_i in ϵ to get

$$c_{\mu_i i} = \epsilon^{\epsilon^{\mu_1\mu_2\cdots\mu_n}} \prod_{j \neq i} A_{\mu_j j} \tag{14}$$

I tested this for some components of 3×3 and 4×4 matrices and it passed.
Going Greek

$$c_{\mu_\alpha \alpha} = \epsilon^{\epsilon^{\mu_1\mu_2\cdots\mu_n}} \prod_{\beta \neq \alpha} A_{\mu_\beta \beta} \tag{15}$$

We know that the inverse tensor can be written in terms of its cofactors and determinant as

$$|A|\bar{A}^{ij} = c^{ji} \tag{16}$$

Using (15) we get

$$\boxed{|A|\bar{A}^{\alpha\mu_\alpha} = \epsilon^{\mu_1\mu_2\mu_3\cdots\mu_n} \prod_{\beta \neq \alpha} A_{\mu_\beta \beta}} \tag{17}$$

which is (1).

On the next page I test that for a rank 3 tensor.

Test that formula

In 3 dimensions (17) it is

$$|A|\bar{A}^{\alpha\mu_\alpha} = \epsilon^{\mu_1\mu_2\mu_3}\prod_{\beta\neq\alpha} A_{\mu_\beta\beta} \tag{18}$$

We'll check that on an example 3×3 matrix / tensor

$$A = A_{ij} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \tag{19}$$

First we use the component-wise inverse matrix formula (16) which is

$$|A|\bar{a}_{ij} = c_{ji} \tag{20}$$

There is no need to calculate the determinant and we have

$$|A|\bar{a}_{ij} = \begin{bmatrix} ei - fh & ch - bi & bf - ce \\ fg - di & ai - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{bmatrix} \tag{21}$$

Now using (18)

$$|A|\bar{A}^{11} = \epsilon^{1\mu_2\mu_3}\prod_{\beta\neq 1} A_{\mu_\beta\beta} = \epsilon^{1\mu_2\mu_3}A_{\mu_2 2}A_{\mu_3 3} = \epsilon^{123}A_{22}A_{33} + \epsilon^{132}A_{32}A_{23} = ei - fh \tag{22}$$

$$|A|\bar{A}^{12} = \epsilon^{2\mu_2\mu_3}\prod_{\beta\neq 1} A_{\mu_\beta\beta} = \epsilon^{2\mu_2\mu_3}A_{\mu_2 2}A_{\mu_3 3} = \epsilon^{213}A_{12}A_{33} + \epsilon^{231}A_{32}A_{13} = -bi + ch \tag{23}$$

$$|A|\bar{A}^{13} = \epsilon^{3\mu_2\mu_3}\prod_{\beta\neq 1} A_{\mu_\beta\beta} = \epsilon^{3\mu_2\mu_3}A_{\mu_2 2}A_{\mu_3 3} = \epsilon^{312}A_{12}A_{23} + \epsilon^{321}A_{22}A_{13} = bf - ce \tag{24}$$

$$|A|\bar{A}^{21} = \epsilon^{\mu_1 1\mu_3}\prod_{\beta\neq 2} A_{\mu_\beta\beta} = \epsilon^{\mu_1 1\mu_3}A_{\mu_1 1}A_{\mu_3 3} = \epsilon^{213}A_{21}A_{33} + \epsilon^{312}A_{31}A_{23} = -di + fg \tag{25}$$

$$|A|\bar{A}^{22} = \epsilon^{\mu_1 2\mu_3}\prod_{\beta\neq 2} A_{\mu_\beta\beta} = \epsilon^{\mu_1 2\mu_3}A_{\mu_1 1}A_{\mu_3 3} = \epsilon^{123}A_{11}A_{33} + \epsilon^{321}A_{31}A_{13} = ai - cg \tag{26}$$

$$|A|\bar{A}^{23} = \epsilon^{\mu_1 3\mu_3}\prod_{\beta\neq 2} A_{\mu_\beta\beta} = \epsilon^{\mu_1 3\mu_3}A_{\mu_1 1}A_{\mu_3 3} = \epsilon^{132}A_{11}A_{23} + \epsilon^{231}A_{21}A_{13} = -af + cd \tag{27}$$

$$|A|\bar{A}^{31} = \epsilon^{\mu_1\mu_2 1}\prod_{\beta\neq 3} A_{\mu_\beta\beta} = \epsilon^{\mu_1\mu_2 1}A_{\mu_1 1}A_{\mu_2 2} = \epsilon^{231}A_{21}A_{32} + \epsilon^{321}A_{31}A_{22} = dh - eg \tag{28}$$

$$|A|\bar{A}^{32} = \epsilon^{\mu_1\mu_2 2}\prod_{\beta\neq 3} A_{\mu_\beta\beta} = \epsilon^{\mu_1\mu_2 2}A_{\mu_1 1}A_{\mu_2 2} = \epsilon^{132}A_{11}A_{32} + \epsilon^{312}A_{31}A_{12} = -ah + bg \tag{29}$$

$$|A|\bar{A}^{33} = \epsilon^{\mu_1\mu_2 3}\prod_{\beta\neq 3} A_{\mu_\beta\beta} = \epsilon^{\mu_1\mu_2 3}A_{\mu_1 1}A_{\mu_2 2} = \epsilon^{123}A_{11}A_{22} + \epsilon^{213}A_{21}A_{12} = ae - bd \tag{30}$$

Hallelujah!

This basically comes from my blog: https://www.general-relativity.net/