The conservation of information from black hole radiance has an unsettled history. The initial indications were that information is lost. Yet a formalism of a consistent quantum theory of gravity with information loss is difficult, for such a theory would have to be nonunitary. In a bet recently Hawking conceded to Preskill that information was preserved in black holes. Information would then not be destroyed, but rather scrambled in such as way as to make its retrieval intractably impossible. A tunnelling approach to quantum radiance by Parikh and Wilczek [1] suggests that the process in total has $\Delta S = 0$, but as recently pointed out in [2] this is the case where the black hole and environment are in thermal equilibrium. However, the negative heat capacity of spacetime means that a black hole slightly removed from equilibrium is unstable and will diverge from equilibrium. This is seen with the evaporation of a black hole, where as its entropy $\Delta S_{bh} \rightarrow 0$ its temperature becomes large. Thus the Parikh-Wilczek tunnelling theory appears to be a "measure zero" case.

A tunnelling process involves an action which is not accessible classically. With the case of a simple tunnelling of a particle through a square barrier, this involves an imaginary momentum or action. In the case of a tunnelling of a particle from a black hole $M \rightarrow M + \delta m$, this involves the imaginary part of the action

$$ImS = Im \int_{r_i}^{r_f} p_r dr. \tag{1}$$

The Hamilton equation $\dot{r} = \frac{\partial H}{\partial p}$ permits this to be written as

$$ImS = Im \int_{r_i}^{r_f} \int_{p_i}^{p_i} \frac{dr}{\dot{r}} dH. \tag{2}$$

Along null geodesics the velocity $\dot{r} = \pm 1 + \sqrt{2M/r}$ the action for the classically forbidden path is

$$ImS = -2\pi \int_0^{\delta m} \frac{dr\ dm}{1 + \sqrt{\frac{2M - \delta m}{r}}} = -2\pi\big((M - \delta m)^2 - M^2\big), \tag{3}$$

which defines $-\frac{1}{2}(S_f - S_i)$. The imaginary part of the action gives the tunnelling probability or emission rate as $\Gamma = exp(-2ImS) = exp(\Delta S)$. For a black hole in equilibrium with the environment the entropy remains on average zero, for with every quanta it emits it will on average absorb another from the environment. For $dM = dQ$ as the first law of black hole thermodynamics with $dS = \frac{dQ}{T}$ holds for a reversible process. However, the fluctuations will eventually cause the black hole to diverge from equilibrium, where no matter how small this is it will cause the black hole radiance to diverge, or for the black hole to acquire larger mass so that its mass increases arbitrarily. Any change in the state of the black hole, whether by emission or absorption, perturbs the black hole from equilibrium.

There are a number of physical ways that black hole radiance are presented. A black hole emits a particle since the quanta which make up a black hole have some small but nonzero probability of existing in a region $r > 2M$. Another interpretation is that virtual electron-positron pairs near the event horizon may permit one in the pair to fall into the hole while the other escapes to infinity. This view is equivalent to saying that an electron or positron propagates backward through time from the black hole and is then scattered into the forward direction by the gravity field. A related interpretation has that the creation of a positive mass-energy particle is associated with the creation of a negative mass-energy particle absorbed by the black hole. As a result the black hole's mass is reduced and a particle escapes. In the case of fermions this is in line with Dirac's original idea of the anti-particle with a negative mass-energy. In all of these cases there is a superposition principle at work. Quanta within the black hole are correlated with quanta in the exterior region. How these quanta are correlated is the fundamental issue. The imaginary action is a measure of the nonlocal correlation a particle in the black hole has with the outside world. In the case of equilibrium with $TdS = dM$ the black hole exchanges entropy with the environment so that the total information of the black hole and environment remains the same. Yet in general the radiance of a black hole will heat up the environment so that $dS > \frac{dM}{T}$, and the same is the case of the black hole absorbs mass-energy.

A black hole will absorb and emit observables, where if information is preserved these observables will have a corollation. The corollation will reflect a quantum process which is unitary, or that the emitted observables are nonlocally entangled with the black hole states in such as way as to preserve information. If information is preserved by a black hole, then in principle a black hole is an efficient teleporter of quantum information. Obviously the quantum gravity evolution must contain some error correction processing which compensates for randomizing quantum fluctuations near the Planck scale. This will be discussed towards the end. At this stage the black hole is shown to ultimately preserve quantum information even for the case that $dS > \frac{dM}{T}$.

The standard "Alice and Bob" problem is considered. Alice has the set of observables A, which she communicates to Bob in a string $x_1x_2\dots x_n$. Bob similarly has the string $y_1y_2\dots y_n$. The von Neumann information each possesses is then $S(X) = -Tr(\rho_X\ log_2\rho_X$, for X either A or B. The entropy of each is then a compartmentalization on ρ_{AB} for the total quantum information both possess and $\rho_A = Tr_B(\rho_{AB})$ and $\rho_B = Tr_A(\rho_{AB})$. Here the trace is over the part of the Hilbert space for Alice or Bob to project out the density operator for Bob or Alice. The density operator ρ_{AB} then defines the joint entropy

$$S(AB) = -Tr(\rho_{AB}\ log_2\rho_{AB}). \tag{4}$$

If Alice transmits her string $x_1x_2\dots x_n$ to Bob this defines the conditional information or entropy $S(A|B)$ as the information communicated by Alice given that Bob has $y_1\dots y_n$ defined as

$$S(A|B) = S(AB) - S(B). \tag{5}$$

If Alice sends this string into a black hole, this is the entropy measured by Bob as measured by the quantum information the black hole emits. The conditional entropy may be defined by a conditional von Neumann entropy definition

$$S(A|B) = -Tr(\rho_B\rho(A|B)\ log_2\rho(A|B) = -Tr(\rho_{AB}log_2\rho_{A|B}), \tag{6}$$

where $\rho_{A|B} = lim_{n\rightarrow\infty}\left(\rho_{AB}{}^{1/n}(\mathbf{1}_A \otimes \rho_B)^{-1/n}\right)^n$. Here $\mathbf{1}_A$ is a unit matrix over the Hilbert space for Alice's quantum information. This means that the entries of $\rho_{A|B}$ can be over unity, which also means that the information content of conditional entropy can be negative as well [3]. Thus quantum information can be negative, in contrast to classical information. The conditional entropy determines how much quantum communication is required to gain complete quantum information of the system in the state ρ_{AB}.

When the conditional entropy is negative Alice can only communicate information about the complete state by classical communication. The sharing of $-S(A|B)$ means that Alice and Bob share an entangled state, which may be used to teleport a state at no entropy cost. The negative quantum information is then the degree of "ignorance" Bob has of the quantum system which cancels out any future information Bob receives. The "hole" that Alice fills in Bob's state ignorance amounts to a merging of her state with Bob's.

The merging of Alice's state with Bob requires the use of an ancillary or reference state. We assume that this ancillary state is the total state of the black hole with the density matrix ρ_{bh}. We then initially have the total state given by $\rho_{ABbh} = \rho_{AB} \otimes \rho_{bh}$. Alice communicates her information so that Bob then holds both his initial state but Alice's as well. Assume that Alice sends n EPR entangled pairs with $nS(A|B)$ to generate the state $\rho_{AA'BB'bh}$ with $S(AA'|BB') < 0$. This addition of an entangled pair reduces the entropy by one q-bit. Thus by $S(A|B) = S(AB) - S(B)$ means that $S(A) > S(AB) = S(bh)$. Thus a measurement of Alice's state projects Bob's state plus that of the black hole into ρ_{Bbh} and the inclusion of the entangled pairs means that the absorption of negative information may put Bob's state into ρ_{ABbh} thus Alice may merge her state with Bob's. The amount of classical information communicated is determined by the mutual information $I(A : bh) = S(bh) + S(A) - S(A|bh)$ between Alice and the black hole. Alice's state is then a product with ρ_{bh}, and the quantum information is the minimal entropy production involved in a measurement. Here the irreversible entropy production is not due to the black hole, but the measurement process.

Consider Bob receiving information from the black hole. The black hole does this through the emission of a particle in some state. Bob measure the change in the black hole entropy according to the information $S(A) < S(AB)$, which is greater than $S(AB) = S(bh)$ The existence of negative entropy may be explicitly demonstrated for the black hole. Let a_k and b_k be the lowering operator for outgoing and ingoing bosonic fields with wave number k on the flat spacetime vacuum $|0\rangle$. These are related to the operators A_k and B_k which annihilate the vacuum outside and inside the black hole, with a Minkowski state $|\,\rangle_m$. The flat space operators are related by the Bogoliubov transformation [4]

$$A_k = \alpha a_k - \beta b_{-k},\ B_k = \alpha b_k - \beta a_{-k}. \tag{7}$$

Here

$$\alpha^2 = cosh^2(g) = \frac{1}{1 - e^{k/T}},\ \beta^2 = sinh^2(g) = \frac{1}{e^{-k/T} - 1}, \tag{8}$$

for g the surface gravity at the horizon determined by the Killing vector field ξ^μ as

$\nabla^\nu(\xi^\mu\xi_\mu) = -2g\xi^\nu$. Transformations $A_k = e^{iH}a_k e^{-iH}$, $B_k = e^{iH}b_k e^{-iH}$ are given by the Hamiltonian

$$H = ig\big(a_k^\dagger b_{-k}^\dagger - a_{-k}b_k + a_{-k}^\dagger b_k^\dagger - a_k b_{-k}\big). \tag{9}$$

The detection of a particle from the black hole, with its mass changing accordingly $M \rightarrow M - \delta M$, and the particle state transitions from $|n_k\rangle \rightarrow |n+1_k\rangle$. The density matrix for the distant external particle state on flat spacetime then changes to $\rho_k' = a_k^\dagger\rho_k a_k$. This is combined with the change in the black hole density of states $\rho_{bh} = \sum_{MM'}|M\rangle\langle M'|$. The black hole density operator changes according to $T_{in}|M\rangle \rightarrow |M + \delta M\rangle$. The black hole adjusts its mass by $M \rightarrow M + \delta M$ if it absorbs the particle, and $M + \delta M \rightarrow M$ if it emits the particle. When the black hole absorbs the particle there is the transition operator T_{in},

$$T_{in}a_k : |M\rangle_{bh}|n_k\rangle_m \rightarrow \sqrt{n}|M + \delta M\rangle_{bh}|n_k - 1\rangle_m, \tag{10}$$

with emission corresponding to

$$T_{out}a_k^\dagger : |M + \delta M\rangle_{bh}|n_k - 1\rangle_m \rightarrow \sqrt{n}|M\rangle_{bh}|n_k\rangle_m. \tag{11}$$

Fields that exit and enter a black hole are coherently correlated, with a scattering operator

$$e^{iH'}a_k e^{-iH'} = \sqrt{1 - P}e^{iH}a_k e^{-iH} + \sqrt{P}e^{iH}b_k e^{-iH}, \tag{12}$$

so that $H' = H + H_s$. P and $1 - P$ give the absorption and emission probabilities of a particle in and out of a black hole. Amplitudes for creation and destruction of a boson are

$$_{bh}\langle M + \delta M|_m\langle n_k - 1|T_{in}a_k|M\rangle_{bh}|n_k\rangle_m = {}_{bh}\langle M + \delta M|T_{in}|M\rangle_{bh}\ {}_m\langle n_k - 1|a_k|n_k\rangle_m$$

$$= (n_k)^{-1/2}{}_{bh}\langle M|T_{in}^\dagger T_{in}|M\rangle_{bh}\ {}_m\langle n_k|(\alpha a_k^\dagger - \beta b_{-k}^\dagger)a_k|n_k\rangle_m \tag{13a}$$

and

$$_{bh}\langle M|_m\langle n_k|T_{out}a_k^\dagger|M + \delta M\rangle_{bh}|n_k - 1\rangle_m = {}_{bh}\langle M|T_{out}|M + \delta M\rangle_{bh}\ {}_m\langle n_k|a_k^\dagger|n_k - 1\rangle_m$$

$$= (n_k)^{-1/2}{}_{bh}\langle M + \delta M|T_{out}T_{out}^\dagger|M + \delta M\rangle_{bh}\ {}_m\langle n_k - 1|(\alpha a_k - \beta b_{-k}^\dagger)a_k^\dagger|n_k - 1\rangle_m. \tag{13b}$$

Consider a normal ordering of $a_k a_k^\dagger$, and matrix elements with $b_{-k}a_k^\dagger$ and $b_{-k}^\dagger a_k$ vanish. The probability for the absorption and emission of a particle are

$$w_{in} = \frac{2\pi}{\hbar}|\rho(M + \delta E)\langle M + \delta M|T_{in}|M\rangle|^2|\langle n_k - 1|a_k|n_k\rangle|^2$$

$$w_{out} = \frac{2\pi}{\hbar}|\rho(M)\langle M|T_{in}|M + \delta M\rangle|^2|\langle n_k|a_k^\dagger n_k - 1\rangle|^2. \tag{14}$$

The Fermi golden rule gives that $w_{out}/w_{in} \;=\; e^{-8\pi M\delta M}$ [4], and with an evaluation of ${}_m\langle\, |a^\dagger_k a_k|\,\rangle_m$ this gives,

$$e^{-8\pi M\delta M} \;=\; \frac{\rho(M)}{\rho(M\,+\,\delta M)}\Big(\frac{(1\,-\,P)n_k\,+\,\beta^2\{1\,+\,(1\,-\,P)n_k\}}{(1\,-\,P)(n_k\,-\,1)\,+\,\beta^2\{1\,+\,(1\,-\,P)(n_k\,-\,1)\}}\Big)^2. \tag{15}$$

Let the ratio of the density of states be $\sim\; PQ(T)$ for $Q(T)\;=\;1/\big(1\,+\,zexp(-8\pi M\delta M)\big)$ the partition function for degrees of freedom on the horizon, with $P\;\simeq\;exp(-8\pi M\delta M)$ the emission probability of a particle of energy δM. $z\;=\;e^{-4\pi M^2}$ is the fugacity of the system. Consider the emission of one particle by the black hole $n_k\;=\;1$, then

$$1\,+\,ze^{-8\pi M\delta M}\;=\;\big((1\,-\,P)\beta^{-2}\,+\,2\,-\,P\big)^2, \tag{16a}$$

so the absorption coefficient is then

$$P\;\simeq\;1\,-\,\frac{1}{2}e^{-4\pi M^2}. \tag{16b}$$

So for quantum unitarity P is nonzero in general.

The density matrix for fields inside and outside the black hole plus the black hole is $\rho_m\;=\;e^{iH}\rho_k\otimes\rho_{bh}e^{-iH}$ then evolves to $\rho'_m\;=\;e^{iH}\rho'_k\otimes\rho'_{bh}e^{-iH}$. For $g\;<<\;1$, or equivalently a black hole sufficiently massive, this gives

$$\begin{aligned}\rho'_m\;&=\;A^\dagger_k T_{out}\rho_m A_k T_{in}\\ &\simeq\;a^\dagger_k T_{out}\rho_m T_{in}a_k\,+\,i\big([H,\,a^\dagger_k]T_{out}\rho_m a_k T_{in}\,+\,a^\dagger_k T_{out}\rho_m[H,\,a_k]T_{in}\big)\\ &=\;a^\dagger_k T_{out}\rho_m T_{in}a_k\,-\,g\big(b_{-k}T_{out}\rho_m T_{in}a_k\,+\,a^\dagger_k T_{out}\rho_m T_{in}b^\dagger_{-k}\big).\end{aligned} \tag{17}$$

The first term gives how the black hole state changes according to the emission of a particle. The remaining two terms indicate a correlation between an ingoing field which enters from the asymptotic flat region with the field emitted by the black hole. These define entangled pairs of states which enter and exit the black hole. If we consider the case where Bob observes all the quanta emitted from the black hole the last two terms consist of fields known to Alice merged with Bob, if we have $S(AB)\;=\;S(bh)$

A measurement of the "in" and "out" states are performed so these states are in an entanglement and define the conditional entropy. Since $\sum_i|i\rangle_{in}$ and $\sum_j|j\rangle_{out}$ are orthogonal the eigenstates of the black hole $|M\rangle$ are also orthogonal to $\sum_j|j\rangle_{out}$. Hence a subset of eigenstates of $|M\rangle$ are those of $\sum_i|i\rangle_{in}$. Further, given the scattering matrix $S\;=\;1\,-\,2\pi iT$ the density matrix for the black hole $\hat\rho_{bh}\;=\;|M\rangle\langle M|$ evolves by unitarity

$$\hat\rho'_{bh}\;=\;S\hat\rho_{bh}S^\dagger. \tag{18}$$

This is still the case for a completely random scattering matrix. This requires that the preservation of quantum information is due to an error correction code. The fidelity of quantum information is necessary in order for quantum gravity to be unitary. The structure of this error correction code is a topic to be introduced in this series.

The von Neumann formula indicates that

$$dS \ \simeq \ k \ ln(2) S d\rho. \tag{19}$$

The differential of the density operator $d\rho$ is entirely hermitian and gives a unitary description for the decay process. It is a simple matter to see that here the conditional entropy $S(B|A) \ = \ S(AB) \ - \ S(A)$ is equal to

$$S(B|A) \ = \ dS \ = \ -gk \ ln(2) S(AB)\big(b_{-k}T_{out}\rho_m T_{in} a_k \ + \ a^\dagger_k T_{out}\rho_m T_{in} b^\dagger_{-k}\big), \tag{20}$$

which is the entangled pair teleported between Alice and Bob. Hence the negative entropy gained by the black hole is accounted for according to negative information, which is further a result of completely unitary processes.

For a black hole in equilibrium $dS \ = \ \frac{dM}{T}$, but this case is not stable. In general it is expected that $dS \ > \ \frac{dM}{T}$, but what is ignored is the negative entropy available. In other words Bob has to option of acquiring additional information for free, with no entropy cost, but ignores it. It is a physics case of winning the lottery, but never checking the ticket numbers. If Bob only considers entropy as classical he will see that the entropy of the black is indeed irreversible. Of course for a large black hole this is what is expected, for the entangled pairs of ingoing and outgoing states are difficult to measure. In other words an outgoing state measured now may be entangled with a state which entered the black hole in the implosion of a star many billions of years earlier. As pointed out in [2] a negative q-bit is equivalent to a q-bit travelling backwards in time. Hence this entangled pair can be thought to involve the merging of a state carrying a negative q-bit which entered the black hole in the past with a state in the future, or equivalently the time reversed connection between of a state which exits the black hole now with a state that entered in the distant past. The nature of quantum black holes and their unitarity and how semi-classical and classical black holes assume a For All Practical Purposes (FAPP) thermodynamic irreversibility is discussed. Here it is found that there is a phase transition effect which differentiates the quantum domain of gravitation from the classical.

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