Infinite summation technique

Contents

List of Figures

Chapter 1

$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^2}{90}$ proved via a contour integral

1.1 Introduction

I first encountered this method many years ago while reading Frederick Reif's "Fundamentals of Statistical and Thermal Physics". In that book, they derived the following result using contour integration:

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}. \tag{1.1}$$

Let us explain the method. It employs the fact that the function

$$\frac{1}{\tan \pi z}$$

has simple poles at all integer values. Indeed, $\tan \pi x$ is linear near n and its derivative there is $+\pi$, so that near n. This suggests that we have the expansion about $z = n$:

$$\frac{1}{\tan \pi z} = \frac{1}{\pi(z-n)} + \cdots \qquad \text{for } n \in \mathbb{Z}. \tag{1.2}$$

This allows us to write

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \sum_{n=1}^{\infty} \frac{1}{2\pi i} \oint_{C_n} \frac{dz}{z^4(z-n)} = \frac{1}{2i} \oint_C \frac{1}{z^4 \tan \pi z} dz \tag{1.3}$$

where the contour C_n is a circle surrounding the point $z = n$ and contour C is defined in fig (a).

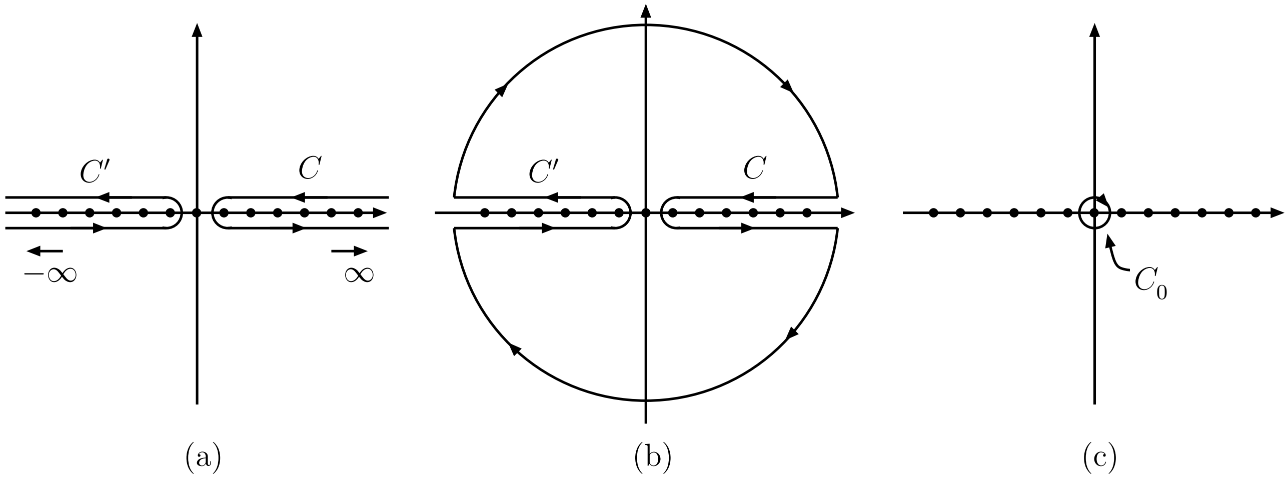

Figure 1.1: .

We have

$$\begin{aligned} 2\sum_{n=1}^{\infty} \frac{1}{n^4} &= \sum_{n=1}^{\infty} \frac{1}{n^4} + \sum_{n=-1}^{-\infty} \frac{1}{n^4} \\ &= \frac{1}{2i} \oint_{C+C'} \frac{1}{z^4 \tan \pi z} dz \end{aligned} \tag{1.4}$$

where the contour C' is defined in fig (a). We complete the path of integration along semicircles at infinity (see fig (b)) since the integral around them will be to zero. Since the resulting enclosed area contains no singularities except at $z = 0$, we can shrink this contour down to an infinitesimal circle C_0 around the origin (see fig (c)). So that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{4i} \oint_{C_0} \frac{dz}{z^4 \tan \pi z} \tag{1.5}$$

We expand the integrand in powers of z about $z = 0$ and isolate the z^{-1} term. We get

$$
\begin{aligned}
\frac{1}{z^4 \tan \pi z} &= \frac{1}{z^4[\pi z + \frac{1}{3}\pi^3 z^3 + \frac{2}{15}\pi^5 z^5 + \cdots]} \\
&= \frac{1}{z^5 \pi[1 + \frac{1}{3}\pi^2 z^2 + \frac{2}{15}\pi^4 z^4 + \cdots]} \\
&= \frac{1}{z^5 \pi}\left[1 - \left(\frac{1}{3}\pi^2 z^2 + \frac{2}{15}\pi^4 z^4 + \cdots\right) + \left(\frac{1}{3}\pi^2 z^2 + \frac{2}{15}\pi^4 z^4 + \cdots\right)^2 - \cdots\right] \\
&= -\frac{2\pi^3}{15z} + \frac{\pi^3}{9z}\cdots \\
&= -\frac{\pi^3}{45z} + \cdots
\end{aligned} \tag{1.6}
$$

Hence (1.5) becomes

$$
\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{4i}(-2\pi i)\left(-\frac{\pi^3}{45}\right) = \frac{\pi^4}{90} \tag{1.7}
$$

giving (1.1).

□

Obviously the technique can be generalised to the case where

$$
\sum_{n=1}^{\infty} \frac{1}{n^{2k}} \tag{1.8}
$$

with

$$
\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{1}{4i}\oint_{C_0} \frac{dz}{z^{2k} \tan \pi z} \tag{1.9}
$$

We could compute the residue using the formula

$$
\mathrm{Res}[f(z)] = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m f(z)] \tag{1.10}
$$

Alternating sums

We can also do sums with alternating sing. For example:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2k}}. \tag{1.11}$$

It employs the fact that the function

$$\frac{1}{\sin \pi z}$$

has simple poles at all integer values but with alternating sign. Indeed, $\sin \pi x$ is linear near n and its derivative there is $(-1)^n \pi$, so that near n. This suggests that we have the expansion about $z = n$:

$$\frac{1}{\sin \pi z} = \frac{(-1)^n}{\pi(z-n)} + \cdots \qquad \text{for } n \in \mathbb{Z}. \tag{1.12}$$

This allows us to write

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2k}} = \sum_{n=1}^{\infty} \frac{1}{2\pi i} \oint_{C_n} (-1)^n \frac{dz}{(z-n)} = \frac{1}{2i} \oint_C \frac{1}{z^4 \sin \pi z} dz \tag{1.13}$$

where the contour C_n is a circle surrounding the point $z = n$ and contour C is defined in fig (a).

The argument goes through the same as before, and we obtain,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2k}} = \frac{1}{4i} \oint_{C_0} \frac{dz}{z^{2k} \sin \pi z}. \tag{1.14}$$

1.1.1 Computing Riemann Zeta function for even integers

The Riemann Zeta functon is defind as follows:

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} \tag{1.15}$$

The first few examples are

$$
\begin{aligned}
\zeta(2) &= \sum_{n=1}^{\infty}\frac{1}{n^2} = (-1)^2(2\pi)^2\frac{B_2}{2(2)!} = \pi^2 B_2 = \frac{\pi^2}{6} = \frac{\pi^2}{6} \\
\zeta(4) &= \sum_{n=1}^{\infty}\frac{1}{n^4} = (-1)^3(2\pi)^4\frac{B_4}{2(4)!} = -\frac{2^3\pi^4}{4!}\pi^2 B_4 = \frac{\pi^4}{90} \\
\zeta(6) &= \sum_{n=1}^{\infty}\frac{1}{n^6} = (-1)^4(2\pi)^6\frac{B_6}{2(6)!} = \frac{2^6\pi^6}{2\cdot 6!}B_6 = \frac{\pi^2}{945} \qquad (1.16)
\end{aligned}
$$

where B_k are the Bernoulli numbers.

Recall

$$
\sum_{n=1}^{\infty}\frac{1}{n^{2k}} = \frac{1}{4i}\oint_{C_0}\frac{dz}{z^{2k}\tan\pi z} \qquad (1.17)
$$

$$
\mathrm{Res}[f(z)] = \frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)] \qquad (1.18)
$$

So that

$$
\sum_{n=1}^{\infty}\frac{1}{n^{2k}} = \frac{1}{4i}\oint_{C_0}\frac{dz}{z^{2k}\tan\pi z} = \frac{1}{4i}(-2\pi i)\frac{1}{(2k)!}\frac{1}{\pi}\lim_{z\to 0}\frac{d^{2k}}{dz^{2k}}\left[\frac{\pi z}{\tan\pi z}\right] \qquad (1.19)
$$

The generating function for the Bernoulli numbers is $\dfrac{x}{e^x-1} = \sum_{n=0}^{\infty}\dfrac{B_n}{n!}x^n$. We have

$$
\begin{aligned}
\frac{x}{2}\coth\frac{x}{2} &= = \frac{x}{2}\frac{e^{\frac{x}{2}}+e^{-\frac{x}{2}}}{e^{\frac{x}{2}}-e^{-\frac{x}{2}}} \\
&= \frac{x}{2}\frac{e^x+1}{e^x-1} = \frac{x}{e^x-1}+\frac{x}{2} = \frac{x}{2}+\sum_{n=0}^{\infty}\frac{B_n}{n!}x^n \qquad (1.20)
\end{aligned}
$$

Using $B_0 = 1$, $B_1 = -\frac{1}{2}$, and that $B_{2k+1} = 0$ for $k \geq 1$, we have

$$
\frac{x}{2}\coth\frac{x}{2} = \sum_{n=0}^{\infty}\frac{B_{2n}}{(2n)!}x^{2n} \qquad (1.21)
$$

That $x\cot(x) = (ix)\coth(ix)$ implies:

$$x \cot x = \sum_{n=0}^{\infty} \frac{4^n(-1)^n B_{2n}}{(2n)!} x^{2n} \tag{1.22}$$

So that

$$\pi z \cot \pi z = \sum_{n=0}^{\infty} \frac{(-1)^n (2\pi)^{2n} B_{2n}}{(2n)!} z^{2n} \tag{1.23}$$

Utilizing the relationship expressed by Equation (1.23), either through the process of dividing it by πz^{2k+1} and extracting the coefficient of z^{-1}, or by using this in (1.19), gives the result:

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = (-1)^{k+1} (2\pi)^{2k} \frac{B_{2k}}{2(2k)!}. \tag{1.24}$$

This is the final result for the Riemann Zeta function for even integers. Up to now, there is no closed form for odd integers known to man (our complex contour integral techniques cannot be applied to the case of odd integers).

We have the Dirichlet eta function

$$\eta(2k) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{2k}} \tag{1.25}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2k}} = \frac{1}{4i} \oint_{C_0} \frac{dz}{z^{2k} \sin \pi z} \tag{1.26}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2k}} = \frac{1}{4i} \oint_{C_0} \frac{dz}{z^{2k} \sin \pi z} = \frac{1}{4i} (-2\pi i) \frac{1}{(2k)!} \lim_{z \to 0} \frac{d^{2k}}{dz^{2k}} \left[\frac{z}{\sin \pi z} \right] \tag{1.27}$$

The Dirichlet eta function is related to the zeta function:

$$\eta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} - 2 \sum_{n=1}^{\infty} \frac{1}{(2n)^k} = \frac{2^{k-1} - 1}{2^{k-1}} \zeta(k) \tag{1.28}$$

Comparing (1.27) and (1.19) and from (1.22), we have an expression for $\csc(x)$ in terms of the Bernoulli numbers:

$$x \csc x = \sum_{n=0}^{\infty} \frac{2(2^{2n-1}-1)(-1)^{n+1} B_{2n}}{(2n)!} x^{2n} \tag{1.29}$$

1.2 Summing over odd integes - alternting sums

1.2.1 Liebniz formula for π

We'll evaluate the sum:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \tag{1.30}$$

We consider the sum

$$\sum_{k=1}^{\infty} \frac{(-1)^k \sin\dfrac{\pi k}{2}}{k} = -\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} \tag{1.31}$$

I will express the sum via a complex contour integration. It employs the fact that the function

$$\frac{\sin\dfrac{\pi z}{2}}{\sin \pi z} \tag{1.32}$$

has simple poles at all integer values except when $\sin\dfrac{\pi z}{2} = 0$.

This allows us to write

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(-1)^k \sin\dfrac{\pi k}{2}}{k} &= \frac{1}{4i} \oint_{C_0} \frac{\sin\dfrac{\pi z}{2}}{z \sin \pi z} dz \\ &= \frac{1}{8i} \oint_{C_0} \frac{1}{z \cos\dfrac{\pi z}{2}} dz \end{aligned} \tag{1.33}$$

where the contour C_0 is defined in fig (c).

$$\begin{aligned} \frac{1}{z\cos\frac{\pi z}{2}} &= \frac{1}{z\left(1+\frac{1}{2!}\frac{\pi^2 z^2}{4}+\cdots\right)} \\ &= \frac{1}{z}\left(1+\frac{\pi^2 z^2}{8}+\cdots\right) \\ &= \frac{1}{z}+\cdots \end{aligned} \tag{1.34}$$

So that

$$\sum_{k=1}^{\infty}\frac{(-1)^k\sin\frac{\pi k}{2}}{k} = \frac{1}{8i}\oint_C \frac{1}{z\cos\frac{\pi z}{2}}dz = \frac{1}{8i}(-2\pi i) = -\frac{\pi}{4} \tag{1.35}$$

So that

$$\frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots \tag{1.36}$$

This is the well know Liebniz forumla.

□

We'll evaluate the sum:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3} \tag{1.37}$$

$$\begin{aligned} \frac{1}{z^3\cos\frac{\pi z}{2}} &= \frac{1}{z^3\left(1+\frac{1}{2!}\frac{\pi^2 z^2}{4}+\cdots\right)} \\ &= \frac{1}{z^3}\left(1+\frac{\pi^2 z^2}{8}+\cdots\right) \\ &= \frac{1}{z^3}+\frac{\pi^2}{8z}+\cdots \end{aligned} \tag{1.38}$$

So that

$$-\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3} = \frac{1}{8i} \oint_C \frac{1}{z^3 \cos \frac{\pi z}{2}} dz = \frac{1}{8i}(-2\pi i)\frac{\pi^2}{8} = -\frac{\pi^2}{32} \tag{1.39}$$

and so

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3} = \frac{\pi^2}{32}. \tag{1.40}$$

Example

Considr the sum

$$f(k) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^{2k+1}} \tag{1.41}$$

for $k = 0, 1, 2, \ldots$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^{2k+1}} = \frac{1}{8i} \oint_C \frac{1}{z^{2k+1} \cos \frac{\pi z}{2}} dz \tag{1.42}$$

$$\mathrm{Res}[f(z)] = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)] \tag{1.43}$$

So that

$$-\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^{2k+1}} = \frac{1}{4i} \oint_{C_0} \frac{dz}{z^{2k+1} \cos \frac{\pi z}{2}} = \frac{1}{4i}(-2\pi i) \frac{1}{(2k)!} \lim_{z \to 0} \frac{d^{2k}}{dz^{2k}} \left[\frac{1}{\cos \frac{\pi z}{2}} \right] \tag{1.44}$$

There is the expansion

$$\sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n} \tag{1.45}$$

where E_n, is the nth Euler number. So that

$$\sec \frac{\pi z}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi)^{2n} E_{2n}}{4^{2n} (2n)!} z^{2n} \tag{1.46}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^{2k+1}} = -\frac{1}{8i}(-2\pi i)\frac{1}{(2k)!}\lim_{z\to 0}\frac{d^{2k}}{dz^{2k}}\left[\frac{1}{\cos\frac{\pi z}{2}}\right] = \frac{(-1)^k(\pi)^{2k+1}E_{2k}}{(2k)!4^{k+1}} \tag{1.47}$$

The first three Euler numbers are $E_0 = 1$, $E_2 = -1$, $E_4 = 5$. So, the first few examples are

$$\begin{aligned} f(0) &= \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)} = \frac{\pi}{4}E_0 = \frac{\pi}{4} \\ f(1) &= \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^3} = (-1)\pi^3\frac{E_2}{(2!)4^2} = \frac{\pi^3}{32} \\ f(2) &= \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^5} = (-1)^2\pi^5\frac{E_4}{(4)!4^3} = \frac{\pi^5}{1536}E_4 = \frac{5\pi^5}{1536} \end{aligned} \tag{1.48}$$

1.3 Summing over odd integes - non-alternting sums

We'll evaluate the sum:

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{2k}} \tag{1.49}$$

We consider the sum

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{1}{(2n-1)^{2k}} &= \sum_{n=1}^{\infty}\frac{1}{n^k} - \sum_{n=1}^{\infty}\frac{1}{(2n)^{2k}} \\ &= \left(1-\frac{1}{2^{2k}}\right)\sum_{n=1}^{\infty}\frac{1}{n^{2k}}. \end{aligned} \tag{1.50}$$

For more general computations we can use the following

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{2k}} &= -\sum_{n=1}^{\infty}(-1)^n\frac{\sin^2\dfrac{\pi n}{2}}{n^k} \\
&= -\frac{1}{4i}\oint_{C_0}\frac{\sin^2\dfrac{\pi z}{2}}{z^{2k}\sin\pi z}dz \\
&= -\frac{1}{8i}\oint_{C_0}\frac{1}{z^{2k-1}z\cot\dfrac{\pi z}{2}}dz \\
&= -\frac{1}{8i}(-2\pi i)\frac{1}{(2k-1)!}\lim_{z\to 0}\frac{d^{2k-1}}{dz^{2k-1}}\tan\frac{\pi z}{2}
\end{aligned} \tag{1.51}
$$

comparing this with (1.50), we obtain

$$
\tan\frac{\pi z}{2} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}2(2^{2n}-1)\pi^{2n-1}B_{2n}}{(2n)!}z^{2n-1}. \tag{1.52}
$$

1.4 Generalising

1.4.1 Additional poles away from the origin

Example:

A simple example of where there are simple poles aside from the origin. Consider the sum

$$
\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}. \tag{1.53}
$$

$$
\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1} = \frac{1}{4i}\oint_{C_s}\frac{dz}{(z^2+1)\sin\pi z} \tag{1.54}
$$

where C_s denotes the contours around all simple poles.

We will need to evalute from the polos $z = 0, i, -i$. The residue at $z = 0$:

$$
\begin{aligned}
\frac{1}{(z^2+1)\sin\pi z} &= \left(1-z^2+\cdots\right)\left(\frac{1}{\pi z(1-\frac{1}{6}\pi^2z^2+\cdots)}\right) \\
&= \left(1-z^2+\cdots\right)\frac{1}{\pi z}\left(1-\frac{1}{6}\pi^2z^2+\cdots\right) \\
&= \frac{1}{\pi z}+\cdots
\end{aligned} \tag{1.55}
$$

So that

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1} &= \frac{1}{4i}(-2\pi i)\left(\frac{1}{\pi}+\frac{1}{2i\sin\pi i}+\frac{1}{-2i\sin-\pi i}\right) \\
&= \frac{1}{2}\left(\frac{\pi}{\sinh\pi}-1\right)
\end{aligned} \tag{1.56}
$$

Example:

Consider the sum

$$
\sum_{n=1}^{\infty}\frac{(-1)^n}{9n^2-4}. \tag{1.57}
$$

The $\dfrac{(-1)^n}{4z^2-9}$ has simple poles at $z=-3/2,3/2$. It has simple poles on the real axis. We can deform the contour as shown if figure . Again we see that evaluating the sum amounts to performing

$$
\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-9}=\frac{1}{4i}\oint_{C_0}\frac{dz}{(4z^2-9)\sin\pi z} \tag{1.58}
$$

We will need to evalute from the polos $z=0,3/2,-3/2$. The residue at $z=0$:

$$
\begin{aligned}
\frac{1}{(4z^2-9)\sin\pi z} &= -\frac{1}{9}\left(1+\frac{4}{9}z^2+\cdots\right)\left(\frac{1}{\pi z(1-\frac{1}{6}\pi^2z^2+\cdots)}\right) \\
&= -\frac{1}{9}\left(1+\frac{4}{9}z^2+\cdots\right)\frac{1}{\pi z}\left(1-\frac{1}{6}\pi^2z^2+\cdots\right) \\
&= -\frac{1}{9\pi z}+\cdots
\end{aligned} \tag{1.59}
$$

So that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-9} &= \frac{1}{4i}(-2\pi i)\left(-\frac{1}{9\pi}+\frac{2}{13\sin\frac{3\pi}{2}}+\frac{2}{-13\sin-\frac{3\pi}{2}}\right)\\ &= \frac{\pi}{18}+\frac{4}{13}. \qquad (1.60)\end{aligned}$$

1.4.2 Generalising summing over symmetric function - alternating sums

The expression (3.5) can be used for general sums of the form

$$\sum_{n=1}^{\infty}(-1)^n f(n) \qquad (1.61)$$

where f is an even function. Viz

$$\sum_{n=1}^{\infty}(-1)^n f(n) = -\frac{1}{8i}(-2\pi i)\sum_i \frac{1}{(m_i-1)!}\lim_{z\to z_i}\frac{d^{m_i-1}}{dz^{m_i-1}}\left[(z-z_i)^{m_i}\frac{f(z)}{\sin\pi z}\right]. \qquad (1.62)$$

where m_i is the order of the pole at $z = m_i$.

1.4.3 Generalising summing over odd integers of even function - non-alternating sums

The expression (1.52) can be used for general sums of the form

$$\sum_{n=1}^{\infty} f(2n-1) \qquad (1.63)$$

where f is an even function.

$$\sum_{n=1}^{\infty} f(2n-1) = -\frac{1}{8i}(-2\pi i)\sum_i \frac{1}{(m_i-1)!}\lim_{z\to z_i}\frac{d^{m_i-1}}{d^{m_i-1}}\left[(z-z_i)^{m_i} f(z)\tan\frac{\pi z}{2}\right]. \qquad (1.64)$$

where m_i is the order of the pole at $z = m_i$.

1.5 Derivation of partial fraction expansions for trigonometric functions

It isn't just the infinite sum of constants we can evaluate, we can also evaluate the infinite sum of a sequence of functions. As a first example of this, we derive the following partial fraction expansions for trigonometric functions:

(a)

$$\pi \cot \pi x = \frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{1}{x^2 - n^2} \tag{1.65}$$

(b)

$$\pi \csc \pi x = \sum_{n=1}^{\infty} \frac{(-1)^n}{x+n} = \frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{(-1)^n}{x^2 - n^2} \tag{1.66}$$

(c)

$$\pi^2 \csc^2 \pi x = \sum_{n=1}^{\infty} \frac{1}{(x+n)^2} \tag{1.67}$$

(d)

$$\pi \sec \pi x = \sum_{n=0}^{\infty} (-1)^n \frac{2n+1}{(n+\frac{1}{2})^2 - x^2} \tag{1.68}$$

(e)

$$\pi \tan \pi x = 2x \sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2})^2 - x^2} \tag{1.69}$$

Proof of (a):

$$\begin{aligned}
\frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{1}{x^2 - n^2} &= \frac{1}{x} + 2x \frac{1}{4i} \oint_{C_s} \frac{dz}{(x^2 - z^2) \tan \pi z} \\
&= \frac{1}{x} + 2x \times \frac{1}{4i} (-2\pi i) \left[\frac{1}{-x^2 \pi} - \frac{1}{2x \tan \pi x} - \frac{1}{-2x \tan \pi(-x)} \right] \\
&= \pi \cot \pi x. \qquad (1.70)
\end{aligned}$$

$\square$

Proof of (b):

Same as (a) but with $\sin \pi z$ instead of $\tan \pi z$.

$\square$

Proof of (c):

$$\begin{aligned}
\frac{1}{x^2} + \sum_{n\neq 0}^{\infty} \frac{1}{(x+n)^2} &= \frac{1}{x^2} + \frac{1}{2i} \oint_{C_s} \frac{dz}{(x^2 - z^2)\tan \pi z} \\
&= \frac{1}{x^2} + \frac{1}{2i}(-2\pi i) \left[\frac{1}{\pi x^2} + \lim_{z\to -x} \frac{d}{dz} \frac{1}{\tan \pi z} \right] \\
&= -\pi \lim_{z\to -x} \frac{d}{dz} \frac{1}{\tan \pi z} \\
&= \pi^2 \csc^2 \pi x. \qquad (1.71)
\end{aligned}$$

$\square$

Proof of (d):

$$\begin{aligned}
\sum_{n=0}^{\infty} (-1)^n \frac{2n+1}{(n+\frac{1}{2})^2 - x^2} &= 4 \sum_{n=0}^{\infty} (-1)^n \frac{2n+1}{(2n+1)^2 - (2x)^2} \\
&= -4 \sum_{n=0}^{\infty} (-1)^n \frac{n \sin \frac{\pi n}{2}}{n^2 - (2x)^2} \\
&= -4 \frac{1}{4i} \oint_{C_s} \frac{z \sin \frac{\pi z}{2}}{[z^2 - (2x)^2] \sin \pi z} dz \\
&= -4 \frac{1}{8i} \oint_{C_s} \frac{z}{[z^2 - (2x)^2] \cos \frac{\pi z}{2}} dz \\
&= -\frac{1}{2i}(-2\pi i) \left[\frac{2x}{4x \cos \pi x} + \frac{-2x}{-4x \cos \pi(-x)} \right] \\
&= \pi \sec \pi x. \qquad (1.72)
\end{aligned}$$

$\square$

Proof of (e):

By part (a), we have

$$
\begin{aligned}
2x\sum_{n=1}^{\infty}\frac{1}{n^2-x^2} &= \frac{1}{x}-\pi\cot\pi x \\
4x\sum_{n=1}^{\infty}\frac{1}{n^2-(2x)^2} &= \frac{1}{2x}-\pi\cot 2\pi x \qquad (1.73)
\end{aligned}
$$

We use these in the following

$$
\begin{aligned}
2x\sum_{n=0}^{\infty}\frac{1}{(n+\frac{1}{2})^2-x^2} &= 8x\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2-(2x)^2} \\
&= 8x\sum_{n=1}^{\infty}\frac{1}{n^2-(2x)^2}-8x\sum_{n=1}^{\infty}\frac{1}{(2n)^2-(2x)^2} \\
&= 8x\sum_{n=1}^{\infty}\frac{1}{n^2-(2x)^2}-8x\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2-x^2} \\
&= 2\left(\frac{1}{2x}-\pi\cot 2\pi x\right)-\left(\frac{1}{x}-\pi\cot\pi x\right) \\
&= \pi\left(\cot\pi x-2\cot 2\pi x\right) \\
&= \pi\left(\cot\pi x-2\frac{\cos^2\pi x-\sin^2\pi x}{2\sin\pi x\cos\pi x}\right) \\
&= \pi\tan\pi x. \qquad (1.74)
\end{aligned}
$$

Or,

$$
\begin{aligned}
2x\sum_{n=0}^{\infty}\frac{1}{(n+\frac{1}{2})^2-x^2} &= 8x\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2-(2x)^2} \\
&= -8x\frac{1}{8i}\oint_{C_s}\frac{1}{[z^2-(2x)^2]\cot\dfrac{\pi z}{2}}dz \\
&= -\frac{x}{i}\oint_{C_s}\frac{\tan\dfrac{\pi z}{2}}{z^2-(2x)^2}dz \\
&= -\frac{x}{i}(-2\pi i)\left[\frac{\tan\pi x}{4x}+\frac{\tan\pi(-x)}{-4x}\right] \\
&= \pi\tan\pi x. \qquad (1.75)
\end{aligned}
$$

$\square$

1.6 Some infinite products

We have the well known formula:

$$\sin x = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right) \tag{1.76}$$

Proof:

From the partial fracton expansion of $cot\pi x$, we have:

$$\begin{aligned}\frac{\cos x}{\sin x} &= \frac{1}{x}+2x\sum_{n=1}^{\infty}\frac{1}{x^2-n^2\pi^2} \\ &= \frac{1}{x}+\sum_{n=1}^{\infty}\frac{-\dfrac{2x}{n^2\pi^2}}{1-\dfrac{x^2}{n^2\pi^2}}\end{aligned} \tag{1.77}$$

From which

$$\frac{d}{dx}\sin x = \frac{d}{dx}\ln x + \sum_{n=1}^{\infty}\frac{d}{dx}\ln\left(1-\frac{x^2}{n^2\pi^2}\right) \tag{1.78}$$

implyng

$$\sin x = Cx\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right) \tag{1.79}$$

That $\lim_{x\to 0}\sin x/x = 1$, implies $C=1$.

$\square$

There is another well known product

$$\cos x = \prod_{n=1}^{\infty}\left(1-\frac{x^2}{(n-1/2)^2\pi^2}\right) \tag{1.80}$$

Proof:

From the partial fracton expansion of $\tan\pi x$, we have:

$$
\begin{aligned}
-\frac{\sin x}{\cos x} &= -2x\sum_{n=1}^{\infty}\frac{1}{(n-1/2)^2\pi^2-x^2} \\
&= \sum_{n=1}^{\infty}\frac{-\dfrac{2x}{(n-1/2)^2\pi^2}}{1-\dfrac{x^2}{(n-1/2)^2\pi^2}}
\end{aligned}
\tag{1.81}
$$

From which

$$
\frac{d}{dx}\cos x = \sum_{n=1}^{\infty}\frac{d}{dx}\ln\left(1-\frac{x^2}{n^2\pi^2}\right) \tag{1.82}
$$

implyng

$$
\cos x = C\prod_{n=1}^{\infty}\left(1-\frac{x^2}{(n-1/2)^2\pi^2}\right) \tag{1.83}
$$

That $\lim_{x\to 0}\cos x = 1$, implies $C = 1$.

□

We can use our summation trick can sometimes be used to evalute a given product.

Example:

$$
f(x) = x^l\prod_{n=1}^{\infty}\left(1-\frac{x^4}{n^4\pi^4}\right). \tag{1.84}
$$

Let us take the logarithm and the take the derivative

$$
\frac{d}{dx}\ln f(x) = \frac{l}{x} + \prod_{n=1}^{\infty}\frac{d}{dx}\ln\left(1-\frac{x^4}{n^4\pi^4}\right) \tag{1.85}
$$

which converts it into a sum

$$
\frac{f'(x)}{f(x)} = \frac{1}{x^l} + \sum_{n=1}^{\infty}\left(\frac{-\dfrac{4x^3}{n^4\pi^4}}{1-\dfrac{x^4}{n^4\pi^4}}\right) \tag{1.86}
$$

which simplifies to

$$\frac{f'(x)}{f(x)} = \frac{l}{x} - 4x^3 \sum_{n=1}^{\infty} \left(\frac{1}{n^4\pi^4 - x^4} \right) \tag{1.87}$$

From which we have

$$\pi \frac{f'(\pi x)}{f(\pi x)} = \frac{l}{x} - 4x^3 \sum_{n=1}^{\infty} \left(\frac{1}{n^4 - x^4} \right) \tag{1.88}$$

Which can be written as a complex contour integral:

$$\frac{d}{dx} \ln f(\pi x) = \frac{l}{x} - 4x^3 \frac{1}{4i} \oint_{C_s} \frac{dz}{(z^4 - x^4) \tan \pi z} \tag{1.89}$$

There are poles at $z = x, xe^{i\pi/2}, xe^{i\pi}, xe^{i\pi 3/2}$. $z^4 - x^4 = (z-x)(z - xe^{i\pi/2})(z - e^{i\pi}x)(z - e^{i\pi 3/4}x)$. We evaluate (1.89) to obtain

$$\begin{aligned} \frac{d}{dx} \ln f(\pi x) &= \frac{l}{x} - 4x^3 \frac{1}{4i}(-2\pi i) \left[\frac{1}{-x^4\pi} + \frac{1}{2x^3(1-e^{i\pi/2})(1-e^{i\pi 3/2}) \tan \pi x} \right. \\ &\quad + \frac{1}{2x^3 e^{i\pi/2}(e^{i\pi/2}-1)(e^{i\pi/2}-e^{i\pi}) \tan \pi(e^{i\pi/2}x)} \\ &\quad + \frac{1}{2e^{i\pi}x^3(e^{i\pi}-e^{i\pi/2})(e^{i\pi}-e^{i\pi 3/2}) \tan \pi(e^{i\pi}x)} \\ &\quad \left. + \frac{1}{2e^{i\pi 3/2}x^3(e^{i\pi 3/2}-1)(e^{i\pi 3/2}-e^{i\pi}) \tan \pi(e^{i\pi 3/2}x)} \right] \end{aligned} \tag{1.90}$$

We will choose $l = 2$. Then

$$\begin{aligned} \frac{d}{dx} \ln f(\pi x) &= \frac{\pi}{2} \left[\frac{e^{i\pi}}{\tan \pi(e^{i\pi}x)} + \frac{e^{i\pi/2}}{\tan \pi(xe^{i\pi/2})} \right. \\ &\quad \left. + \frac{e^{i\pi}}{\tan \pi(e^{i\pi}x)} + \frac{e^{i\pi 3/2}}{\tan \pi(e^{i\pi 3/2}x)} \right] \end{aligned} \tag{1.91}$$

We have

$$\frac{d}{dx}\ln f(\pi x) = \frac{1}{2}\left[\frac{d}{dx}\ln\sin|\pi x| + \frac{d}{dx}\ln\sin[e^{i\pi/2}|\pi x|] + \frac{d}{dx}\ln\sin[e^{i\pi}|\pi x|] + \frac{d}{dx}\ln\sin[e^{i\pi 3/2}|\pi x|]\right] \tag{1.92}$$

So that

$$f(\pi x) = C\exp\left\{\frac{1}{2}\left[\ln\sin|\pi x| + \ln\sin e^{i\pi/2}|\pi x| + \ln\sin e^{i\pi}|\pi x| + \ln\sin e^{i\pi 3/2}|\pi x|\right]\right\} \tag{1.93}$$

and

$$f(\pi x) = C\left[\sin[|\pi x|]\cdot\sin[e^{i\pi}|\pi x|]\cdot\sin[e^{i\pi/2}|\pi x|]\cdot\sin[e^{i\pi 3/2}|\pi x|]\right]^{1/2} \tag{1.94}$$

Using this in (1.84)

$$C\left[\prod_{k=0}^{3}\sin[|x|e^{\frac{i2\pi k}{4}}]\right]^{1/2} = x^2\prod_{n=1}^{\infty}\left(1-\frac{x^4}{n^4\pi^4}\right) \tag{1.95}$$

Using $\lim_{x\to 0}\sin[Ax]/x = A$, we have

$$C = \sqrt{\prod_{k=0}^{3} e^{\frac{i2\pi k}{4}}} = \sqrt{e^{i\frac{2\pi}{4}[1+2+3]}} = \sqrt{e^{i3\pi}} = i. \tag{1.96}$$

Therefore

$$\sin x\sinh x = x^2\prod_{n=1}^{\infty}\left(1-\frac{x^4}{n^4\pi^4}\right). \tag{1.97}$$

This can be checked by refering to the tables:

http://www-elsa.physik.uni-bonn.de/∼dieckman/InfProd/InfProd.html

where they state

$$\prod_{k=-\infty}^{\infty}\frac{k^n-a^n}{k^n-b^n} = \prod_{k=0}^{n-1}\frac{\sin[\pi a e^{\frac{i2k\pi}{n}}]}{\sin[\pi b e^{\frac{i2k\pi}{n}}]}. \tag{1.98}$$

$\square$

Example:

Consider the more general case:

$$f(x) = x^l \prod_{n=1}^{\infty} \left(1 - \frac{x^{2k}}{n^{2k}\pi^{2k}}\right). \tag{1.99}$$

Let us take the logarithm and then take the derivative

$$\frac{d}{dx} \ln f(x) = \frac{l}{x} + \prod_{n=1}^{\infty} \frac{d}{dx} \ln \left(1 - \frac{x^{2k}}{n^{2k}\pi^{2k}}\right) \tag{1.100}$$

which converts it into a sum

$$\frac{f'(x)}{f(x)} = \frac{1}{x^l} + \sum_{n=1}^{\infty} \left(\frac{-\frac{2kx^{2k-1}}{n^{2k}\pi^{2k}}}{1 - \frac{x^{2k}}{n^{2k}\pi^{2k}}} \right) \tag{1.101}$$

which simplifies to

$$\frac{f'(x)}{f(x)} = \frac{l}{x} - 2kx^{2k-1} \sum_{n=1}^{\infty} \left(\frac{1}{n^{2k}\pi^{2k} - x^{2k}} \right) \tag{1.102}$$

From which we have

$$\pi \frac{f'(\pi x)}{f(\pi x)} = \frac{l}{x} - 2kx^{2k-1} \sum_{n=1}^{\infty} \left(\frac{1}{n^{2k} - x^{2k}} \right) \tag{1.103}$$

Which can be written as a complex contour integral:

$$\frac{d}{dx} \ln f(\pi x) = \frac{l}{x} - 2kx^{2k-1} \frac{1}{4i} \oint_{C_s} \frac{dz}{(z^{2k} - x^{2k}) \tan \pi z} \tag{1.104}$$

There are poles at $z = x, xe^{i\pi/2k}, xe^{i2\pi/2k}, xe^{i\pi 3/2k}, \ldots, xe^{i\pi(2k-1)/2k}$. Note

$$\lim_{z\to xe^{\frac{i2\pi q}{2k}}}\frac{z^{2k}-x^{2k}}{z-x^{\frac{i2\pi q}{2k}}}=\lim_{z\to xe^{\frac{i2\pi q}{2k}}}\frac{\frac{d}{dz}(z^{2k}-x^{2k})}{\frac{d}{dz}(z-x^{\frac{i2\pi q}{2k}})}=\lim_{z\to xe^{\frac{i2\pi q}{2k}}}2kz^{2k-1}=2kx^{2k-1}e^{-\frac{i2\pi q}{2k}} \quad (1.105)$$

Using this we evaluate (1.104) to obtain

$$\frac{d}{dx}\ln f(\pi x)=\frac{l}{x}-2kx^{2k-1}\frac{1}{4i}(-2\pi i)\left[\frac{1}{-x^{2k}\pi}+\sum_{q=0}^{2k-1}\frac{e^{\frac{i2\pi q}{2k}}}{2kx^{2k-1}\tan\pi(e^{\frac{i2\pi q}{2k}}x)}\right] \quad (1.106)$$

We will choose $l=k$. Then

$$\frac{d}{dx}\ln f(\pi x)=\frac{1}{2}\left[\sum_{q=0}^{2k-1}\frac{d}{dx}\ln\sin[e^{\frac{i2\pi q}{2k}}|\pi x|]\right] \quad (1.107)$$

So that

$$f(\pi x)=C\exp\left\{\frac{1}{2}\left[\sum_{q=0}^{2k-1}\ln\sin[e^{\frac{i2\pi q}{2k}}|\pi x|]\right]\right\} \quad (1.108)$$

and

$$f(\pi x)=C\left[\prod_{q=0}^{2k-1}\sin[e^{\frac{i2\pi q}{2k}}|\pi x|]\right]^{1/2} \quad (1.109)$$

Using this in (1.99)

$$C\left[\prod_{q=0}^{2k-1}\sin[e^{\frac{i2\pi q}{2k}}|x|]\right]^{1/2}=x^k\prod_{n=1}^{\infty}\left(1-\frac{x^{2k}}{n^{2k}\pi^{2k}}\right) \quad (1.110)$$

Using $\lim_{x\to 0}\sin[Ax]/x=A$, we have

$$C = \sqrt{\prod_{q=0}^{2k-1} e^{\frac{i2\pi q}{2k}}} = \sqrt{e^{\frac{i2\pi[1+2+\cdots+(2k-1)]}{2k}}} = \sqrt{e^{i\pi(2k-1)}} = \sqrt{(-1)^{2k-1}} = i. \tag{1.111}$$

Therefore

$$i\left[\prod_{q=0}^{n-1} \sin[e^{\frac{i2\pi q}{2k}}|x|]\right]^{1/2} = x^k \prod_{n=1}^{\infty}\left(1 - \frac{x^{2k}}{n^{2k}\pi^{2k}}\right). \tag{1.112}$$

which is in agreement with (1.98)

□

In actual fact, we can derive (1.98) for even and odd powers of n using the result we have just obtained.

Proof:

For even powers:

$$\begin{aligned}
\prod_{n=-\infty}^{\infty} \frac{n^{2k} - x^{2k}}{n^{2k} - y^{2k}} &= \frac{x^{2k}}{y^{2k}}\left(\prod_{n=1}^{\infty} \frac{n^{2k} - x^{2k}}{n^{2k} - y^{2k}}\right)^2 \\
&= \left(\frac{x^k \prod_{n=1}^{\infty}\left(1 - \dfrac{x^{2k}}{n^{2k}}\right)}{y^k \prod_{n=1}^{\infty}\left(1 - \dfrac{y^{2k}}{n^{2k}}\right)}\right)^2 \\
&= \prod_{q=0}^{2k-1} \frac{\sin[e^{\frac{i2\pi q}{2k}}|x|]}{\sin[e^{\frac{i2\pi q}{2k}}|y|]}
\end{aligned} \tag{1.113}$$

For odd powers:

$$
\begin{aligned}
\prod_{n=-\infty}^{\infty}\frac{n^{2k+1}-x^{2k+1}}{n^{2k+1}-y^{2k+1}} &= \frac{x^{2k+1}}{y^{2k+1}}\prod_{n=1}^{\infty}\frac{n^{2k+1}-x^{2k+1}}{n^{2k+1}-y^{2k+1}}\prod_{n=1}^{\infty}\frac{-n^{2k+1}-x^{2k+1}}{-n^{2k+1}-y^{2k+1}} \\
&= \frac{x^{2k+1}\prod_{n=1}^{\infty}\left(1-\dfrac{x^{4k+2}}{n^{4k+2}}\right)}{y^{2k+1}\prod_{n=1}^{\infty}\left(1-\dfrac{y^{4k+2}}{n^{4k+2}}\right)} \\
&= \sqrt{\frac{\prod_{q=0}^{4k+1}\sin[e^{\frac{i2\pi q}{4k+2}}|x|]}{\prod_{q=0}^{4k+1}\sin[e^{\frac{i2\pi q}{4k+2}}|y|]}} \\
&= \sqrt{\frac{\prod_{q=0}^{2k}\sin[e^{\frac{i2\pi 2q}{4k+2}}|x|]}{\prod_{q=0}^{2k}\sin[e^{\frac{i2\pi 2q}{4k+2}}|y|]}\cdot\frac{\prod_{q=0}^{2k}\sin[e^{\frac{i2\pi(2q+1)}{4k+2}}|x|]}{\prod_{q=0}^{2k}\sin[e^{\frac{i2\pi(2q+1)}{4k+2}}|y|]}} \qquad (1.114)
\end{aligned}
$$

We have

$$
\begin{aligned}
\prod_{q=0}^{2k}\sin[e^{\frac{i2\pi(2q+1)}{4k+2}}|x|] &= \prod_{q=0}^{2k}\sin[e^{\frac{i2\pi(2(2k-q)+1)}{4k+2}}|x|] \\
&= \prod_{q=0}^{2k}\sin[e^{\frac{-i2\pi(q+1)}{2k+1}}|x|] \\
&= \prod_{q=0}^{2k}\sin[e^{\frac{i2\pi(2k+1-q-1)}{2k+1}}|x|] \\
&= \prod_{q=0}^{2k}\sin[e^{\frac{i2\pi q}{2k+1}}|x|] \qquad (1.115)
\end{aligned}
$$

Using this in (1.114) we obtain

$$
\prod_{n=-\infty}^{\infty}\frac{n^{2k+1}-x^{2k+1}}{n^{2k+1}-y^{2k+1}} = \prod_{q=0}^{2k}\frac{\sin[e^{\frac{i2\pi q}{2k+1}}|x|]}{\sin[e^{\frac{i2\pi q}{2k+1}}|y|]} \qquad (1.116)
$$

□

Example

Let us take an example from:

http://www-elsa.physik.uni-bonn.de/∼dieckman/InfProd/InfProd.html of:

$$\prod_{k=1}^{\infty}\left(1-\frac{a}{(k^2+b)^2}\right)=\frac{b}{\sqrt{a-b^2}}\frac{\sinh[\pi\sqrt{\sqrt{a}+b}]\sin[\pi\sqrt{\sqrt{a}-b}]}{\sinh^2[\sqrt{b}\pi]}. \tag{1.117}$$

Proof:

Set

$$f(x,y)=\prod_{n=1}^{\infty}\left(1-\frac{x}{(n^2+y)^2}\right) \tag{1.118}$$

then

$$\begin{aligned}
\frac{\partial}{\partial x}f(x,y) &= \sum_{n=1}^{\infty}\frac{-\dfrac{1}{(n^2+y)^2}}{1-\dfrac{x}{(n^2+y)^2}} \\
&= -\sum_{n=1}^{\infty}\frac{1}{(n^2+y)^2-x} \\
&= -\frac{1}{4i}\oint\frac{dz}{[(z^2+y)^2-x]\tan\pi z} \\
&= -\frac{1}{4i}(-2\pi i)\left[-\frac{1}{(x-y^2)\pi}+\frac{1}{2\sqrt{-y+\sqrt{x}}\sqrt{x}\tan\pi\sqrt{-y+\sqrt{x}}}\right. \\
&\quad \left.-\frac{1}{2\sqrt{-y-\sqrt{x}}\sqrt{x}\tan\pi\sqrt{-y-\sqrt{x}}}\right] \\
&= -\frac{1}{2}\frac{\partial}{\partial x}\ln(x-y^2)+\frac{\partial}{\partial x}\ln\sin[\pi\sqrt{-y+\sqrt{x}}]+\frac{\partial}{\partial x}\ln\sin[\pi\sqrt{-y-\sqrt{x}}]
\end{aligned} \tag{1.119}$$

This implies

$$f(x,y)=\frac{g(y)}{\sqrt{x-y^2}}\sin[\pi\sqrt{-y+\sqrt{x}}]\sin[\pi\sqrt{-y-\sqrt{x}}] \tag{1.120}$$

where $g(y)$ is a yet to be determined function of y. We have

$$1=f(0,y)=\frac{g(y)}{-iy}\sin^2[\pi iy] \tag{1.121}$$

So that

$$f(x,y) = \frac{y}{\sqrt{x-y^2}} \frac{\sinh[\pi\sqrt{\sqrt{x}+y}]\sin[\pi\sqrt{\sqrt{x}-y}]}{\sinh^2[\pi\sqrt{y}]}. \tag{1.122}$$

□

Example

Let us take an example from:

http://www-elsa.physik.uni-bonn.de/∼dieckman/InfProd/InfProd.html

of:

$$\prod_{k=-\infty}^{\infty} \left(1 - \frac{a^n}{(k+b)^n}\right) = \csc[b\pi]^n \prod_{k=0}^{n-1} \sin\left[\pi\left(b - ae^{\frac{i2k\pi}{n}}\right)\right] \tag{1.123}$$

Proof:

Set

$$f(x,y) = \prod_{n=-\infty}^{\infty} \left(1 - \frac{x^k}{(n+y)^k}\right) \tag{1.124}$$

then

$$\begin{aligned}
\frac{\partial}{\partial x} f(x,y) &= \sum_{n=-\infty}^{\infty} \frac{\frac{-kx^{k-1}}{(n+y)^k}}{1 - \frac{x^k}{(n+y)^k}} \\
&= \sum_{n=-\infty}^{\infty} \frac{-kx^{k-1}}{(n+y)^k - x^k} \\
&= \sum_{n\neq 0}^{\infty} \frac{-kx^{k-1}}{(n+y)^k - x^k} + \frac{-kx^{k-1}}{y^k - x^k} \\
&= \frac{-kx^{k-1}}{y^k - x^k} - kx^{k-1} \frac{1}{2i} \oint_{C+C'} \frac{dz}{[(z+y)^k - x^k]\tan\pi z} \\
&= \frac{-kx^{k-1}}{y^k - x^k} - kx^{k-1} \frac{1}{2i} \oint_{C_s} \frac{dz}{[(z+y)^k - x^k]\tan\pi z}
\end{aligned} \tag{1.125}$$

Note that it is not an even function of n. There are poles at $z = -y + xe^{i2\pi/k}, -y + xe^{i2\pi/k}, -y + xe^{i3\pi/k}, \ldots, -y + xe^{i(k-1)\pi/k}$. Note

$$\begin{aligned}
\lim_{z \to -y + xe^{\frac{i2\pi q}{k}}} \frac{(z+y)^k - x^k}{z + y - x^{\frac{i2\pi q}{k}}} &= \lim_{z \to -y + xe^{\frac{i2\pi q}{k}}} \frac{\frac{d}{dz}[(z+y)^k - x^k]}{\frac{d}{dz}(z + y - x^{\frac{i2\pi q}{k}})} \\
&= \lim_{z \to -y + xe^{\frac{i2\pi q}{k}}} k(z+y)^{k-1} = kx^{k-1} e^{-\frac{i2\pi q}{k}} \qquad (1.126)
\end{aligned}$$

Using this we evaluate (1.125) to obtain

$$\frac{\partial}{\partial x} \ln f(x,y) = \frac{-kx^{k-1}}{y^k - x^k} - kx^{k-1} \frac{1}{2i}(-2\pi i) \left[\frac{1}{(y^k - x^k)\pi} + \sum_{q=0}^{k-1} \frac{e^{\frac{i2\pi q}{k}}}{kx^{k-1} \tan \pi (e^{\frac{i2\pi q}{k}} x - y)} \right] \qquad (1.127)$$

Then

$$\frac{\partial}{\partial x} \ln f(x,y) = \sum_{q=0}^{k-1} \frac{\partial}{\partial x} \ln \sin[\pi(e^{\frac{i2\pi q}{k}} x - y)] \qquad (1.128)$$

Implying

$$f(x,y) = g(y) \prod_{q=0}^{k-1} \sin[\pi(e^{\frac{i2\pi q}{k}} x - y)] \qquad (1.129)$$

where $g(y)$ is a yet to be determined function of y. We have

$$1 = f(0,y) = g(y) \prod_{q=0}^{k-1} \sin[\pi y] \qquad (1.130)$$

So that

$$f(x,y) = \csc^k[\pi y] \prod_{q=0}^{k-1} \sin[\pi(y - e^{\frac{i2\pi q}{k}} x)] \qquad (1.131)$$

$\square$

Other examples from:

http://www-elsa.physik.uni-bonn.de/∼dieckman/InfProd/InfProd.html

I suspect can be derived via the above methods include:

$$\prod_{k=1}^{\infty}\left(1+\frac{c}{ak^2+b}\right)=\frac{\sqrt{b+c}/\sqrt{b}}{\sinh\left[\pi\sqrt{b}/\sqrt{a}\right]}\sinh\left[\frac{\pi\sqrt{b+c}}{\sqrt{a}}\right] \tag{1.132}$$

$$\prod_{k=1}^{\infty}\left(1+\frac{(-1)^k c}{ak^2+b}\right)=\frac{2\sqrt{b+c}/\sqrt{b}}{\sinh\left[\pi\sqrt{b}/\sqrt{a}\right]}\cosh\left[\frac{\pi\sqrt{b+c}}{2\sqrt{a}}\right]\sinh\left[\frac{\pi\sqrt{b+c}}{2\sqrt{a}}\right] \tag{1.133}$$

$$\begin{aligned}\prod_{k=1}^{\infty}\left(1-\frac{(-1)^k a}{(k^2+b)^2}\right) &= \frac{b\sinh\left[\frac{\pi}{2}\sqrt{\sqrt{a}+b}\right]\sin\left[\frac{\pi}{2}\sqrt{\sqrt{a}-b}\right]}{2\sinh\left[\frac{\pi\sqrt{b}}{2}\pi\right]^2\cosh\left[\frac{\pi\sqrt{b}}{2}\pi\right]^2}\left[\cosh\left[\pi\sqrt{\frac{\sqrt{a+b^2}+b}{2}}\right]\right.\\ &\quad\left.+\cos\left[\pi\sqrt{\frac{\sqrt{a+b^2}-b}{2}}\right]\right]\end{aligned} \tag{1.134}$$

$$\prod_{k=1}^{\infty}\left(1+\frac{ak^2}{(k^2+b)^2}\right)=\frac{\cos[\pi\sqrt{a-4b}]-\cos[\pi\sqrt{a}]}{2\sinh[\pi\sqrt{b}]^2} \tag{1.135}$$

I leave this for an exercise for the reader.

Maybe the reader can come up with forumla that are not on the list!

1.7 More general summations with additional poles away from the origin

1.7.1 Sums of rational polynomial fractions with even powers of n

Consider the sum

$$\sum_{n=1}^{\infty}\frac{\prod_{j=1}^{M}(n^2-b_j)}{n^{2L}\prod_{i=1}^{N}(n^2-a_i)} \tag{1.136}$$

where $b_i \neq a_j$ for any $i = 1, \ldots, N$ or $j = 1, \ldots, M$ and $a_i \neq 0$ for $i = 1, \ldots, N$. In order for the summation to converge we require $N + 2L > M$.

$$\sum_{n=1}^{\infty} \frac{\prod_{j=1}^{M}(n^2 - b_i)}{n^{2L}\prod_{i=1}^{N}(n^2 - a_i)} = \frac{1}{4i}\oint_{C_s} \frac{\prod_{j=1}^{M}(z^2 - b_j)}{\prod_{i=1}^{N}(z^2 - a_i)z^{2L}\tan \pi z} dz \tag{1.137}$$

The integrand has a simple pole at $z = 0$ and has simple poles at $z = \pm a_i$ for all $i = 1, \ldots, N$. As

$$\frac{P(z)}{z^{2L}Q(z)} = \frac{\prod_{j=1}^{M}(z^2 - b_j)}{z^{2L}\prod_{i=1}^{N}(z^2 - a_i)} \tag{1.138}$$

is a rational proper fraction, it can be decomposed into

$$\begin{aligned} \frac{P(z)}{z^{2L}Q(z)} &= \sum_{i=1}^{N}\left(\frac{A_{i1}}{z - \sqrt{\alpha_i}} + \frac{A_{i2}}{(z - \sqrt{a_i})^2} + \cdots + \frac{A_{ik_i}}{(z - \sqrt{a_i})^{k_i}}\right) + \\ &\quad + \sum_{i=1}^{N}\left(\frac{B_{i1}}{z + \sqrt{\alpha_i}} + \frac{B_{i2}}{(z + \sqrt{a_i})^2} + \cdots + \frac{B_{ik_i}}{(z + \sqrt{a_i})^{k_i}}\right) + \frac{C_1}{z} + \frac{C_2}{z^2} + \cdots \frac{C_{2L}}{z^{2L}} \end{aligned} \tag{1.139}$$

where k_i is the multiplicity of the root $\sqrt{\alpha_i}$ (and the root $-\sqrt{a_i}$). We have

$$A_{i1} = \text{Res}[P/z^{2L}Q, \pm\sqrt{a_i}] \quad \text{and} \quad B_{i1} = \text{Res}[P/z^{2L}Q, \pm\sqrt{a_i}]. \tag{1.140}$$

These are given by the explicit formulas

$$A_{i1} = \frac{1}{(k_i - 1)!}\lim_{z \to \sqrt{a_i}} \frac{d^{k_i - 1}}{dz^{k_i - 1}}\left[(z - \sqrt{a_i})^{k_i}\frac{P(z)}{z^{2L}Q(z)}\right] \tag{1.141}$$

and

$$B_{i1} = \frac{1}{(k_i - 1)!}\lim_{z \to -\sqrt{a_i}} \frac{d^{k_i - 1}}{dz^{k_i - 1}}\left[(z + \sqrt{a_i})^{k_i}\frac{P(z)}{z^{2L}Q(z)}\right] \tag{1.142}$$

We can write down an expression for the sum (1.136):

$$\sum_{n=1}^{\infty} \frac{\prod_{j=1}^{M}(n^2-b_j)}{n^{2L}\prod_{i=1}^{N}(n^2-a_i)} = \frac{\pi}{2}\sum_i \frac{B_{i1}-A_{i1}}{\tan\pi\sqrt{a_i}} + \frac{\prod_{j=1}^{M} b_j}{\prod_{j=1}^{N} a_i}(-1)^{M+N+L+1}(2\pi)^{2L}\frac{B_{2L}}{2(2L)!} \quad (1.143)$$

where we have used (1.23) to evaluate the pole at $z=0$.

When $Q(x)$ has simple roots

The formula is more simple when $Q(x)$ has simple roots. In this case

$$\frac{P(x)}{x^{2L}Q(x)} = \sum_{i=1}^{N}\frac{P(\sqrt{a_i})}{a_i^L Q'(\sqrt{a_i})}\frac{1}{x-a_i} + \sum_{i=1}^{N}\frac{P(-\sqrt{a_i})}{a_i^L Q'(-\sqrt{a_i})}\frac{1}{x+\alpha_i} + \frac{C_1}{x} + \frac{C_2}{x^2} + \cdots \frac{C_{2L}}{x^{2L}} \quad (1.144)$$

and the sum is

$$\sum_{n=1}^{\infty} \frac{\prod_{j=1}^{M}(n^2-b_j)}{n^{2L}\prod_{i=1}^{N}(n^2-a_i)} = -\pi\sum_{i=1}^{N}\frac{P(\sqrt{a_i})}{Q'(\sqrt{a_i})}\frac{1}{a_i^L\tan\pi\sqrt{a_i}} + \frac{\prod_{j=1}^{M} b_j}{\prod_{j=1}^{N} a_i}(-1)^{M+N+L+1}(2\pi)^{2L}\frac{B_{2L}}{2(2L)!} \quad (1.145)$$

1.7.2 Sums of rational polynomial fractions with even powers of n - Alternating sums

Consider the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^n\prod_{i=1}^{j}(n^2-b_i)}{n^{2L}\prod_{i=1}^{j}(n^2-a_i)} \quad (1.146)$$

where $b_i \neq a_j$ for any i or j.

$$\sum_{n=1}^{\infty} \frac{(-1)^n\prod_{j=1}^{M}(n^2-b_j)}{n^{2L}\prod_{i=1}^{N}(n^2-a_i)} = \frac{1}{4i}\oint_{C_s}\frac{\prod_{j=1}^{M}(z^2-b_j)}{\prod_{i=1}^{N}(z^2-a_i)z^{2L}\sin\pi z}dz \quad (1.147)$$

We can write down an expression for the sum (1.146):

$$\sum_{n=1}^{\infty} \frac{(-1)^n\prod_{j=1}^{M}(n^2-b_j)}{n^{2L}\prod_{i=1}^{N}(n^2-a_i)} = \frac{\pi}{2}\sum_i \frac{B_{i1}-A_{i1}}{\sin\pi\sqrt{a_i}} + \frac{\prod_{j=1}^{M} b_j}{\prod_{j=1}^{N} \alpha_i}(-1)^{M+N+L+1}(2^{2L-1}-1)(2\pi)^{2L}\frac{B_{2L}}{(2L)!} \quad (1.148)$$

where we have used (3.5) to evaluate the pole at $z = 0$.

Again, the formula is more simple when $Q(x)$ has simple roots, and the sum is

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^n \prod_{j=1}^{M}(n^2 - b_j)}{n^{2L} \prod_{i=1}^{N}(n^2 - a_i)} &= -\pi \sum_{i=1}^{N} \frac{P(\sqrt{a_i})}{Q'(\sqrt{a_i})} \frac{1}{a_i^L \sin \pi \sqrt{a_i}} \\ &\quad + \frac{\prod_{j=1}^{M} \beta_j}{\prod_{j=1}^{N} \alpha_i} (-1)^{M+N+L+1} (2^{2L-1} - 1)(2\pi)^{2L} \frac{B_{2L}}{(2L)!}\end{aligned} \tag{1.149}$$

Chapter 2

Inverting Fourier series

The technique could be used to sum up a series of functions, and not just constants.

2.1 Reminder of Fourier series

Recall that the formal Fourier series of s is given by

$$s(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nx} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 2\pi nx + \sum_{n=1}^{\infty} b_n \sin 2\pi nx \tag{2.1}$$

Fourier series for interval $-\frac{1}{2} \leq x \leq \frac{1}{2}$

$$s(x) = \sum_{n=1}^{\infty} a_n \cos \frac{2\pi nx}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{2\pi nx}{L} \tag{2.2}$$

Using

$$\begin{aligned}
\int_{-L/2}^{L/2} \cos 2\pi mx \cos 2\pi nx dx &= \begin{cases} \frac{L}{2}\delta_{m,n} & \text{if } m \neq 0 \\ 0 & \text{if } m = 0 \end{cases} \\
\int_{-L/2}^{L/2} \sin 2\pi mx \sin 2\pi nx dx &= \begin{cases} \frac{L}{2}\delta_{m,n} & \text{if } m \neq 0 \\ 0 & \text{if } m = 0 \end{cases} \\
\int_{-L/2}^{L/2} \cos 2\pi mx \sin 2\pi nx dx &= 0 \quad \text{for all } m \text{ and } n.
\end{aligned} \tag{2.3}$$

we have

$$
\begin{aligned}
a_0 &= \frac{1}{L}\int_{-L/2}^{L/2} s(x)dx \\
a_n &= (c_n + c_{-n}) = \frac{2}{L}\int_{-L/2}^{L/2} s(x)\cos\frac{2\pi n x}{L}dx \\
b_n &= i(c_n - c_{-n})\frac{2}{L}\int_{-L/2}^{L/2} s(x)\sin\frac{2\pi n x}{L}dx
\end{aligned} \tag{2.4}
$$

If $s(-x) = s(x)$ then the series contains only cosine terms. If $s(-x) = -s(x)$ then the series contains only sine terms.

Fourier series for interval $0 \leq x \leq 1$

$$
s(x) = \sum_{n=1}^{\infty} a_n \cos\frac{2\pi n x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{2\pi n x}{L} \tag{2.5}
$$

Using

$$
\begin{aligned}
\int_0^L \cos\frac{2\pi m x}{L}\cos\frac{2\pi n x}{L}dx &= \begin{cases} \frac{L}{2}\delta_{m,n} & \text{if } m \neq 0 \\ 0 & \text{if } m = 0 \end{cases} \\
\int_0^L \sin\frac{2\pi m x}{L}\sin\frac{2\pi n x}{L}dx &= \begin{cases} \frac{L}{2}\delta_{m,n} & \text{if } m \neq 0 \\ 0 & \text{if } m = 0 \end{cases} \\
\int_0^L \cos\frac{2\pi m x}{L}\sin\frac{2\pi n x}{L}dx &= 0 \quad \text{for all } m \text{ and } n.
\end{aligned} \tag{2.6}
$$

we have

$$
\begin{aligned}
a_0 &= \frac{1}{L}\int_0^L s(x)dx \\
a_n &= (c_n + c_{-n}) = \frac{2}{L}\int_0^L s(x)\cos\frac{2\pi n x}{L}dx \\
b_n &= i(c_n - c_{-n}) = \frac{2}{L}\int_0^L s(x)\sin\frac{2\pi n x}{L}dx
\end{aligned} \tag{2.7}
$$

If $s(L-x) = s(x)$ then the series contains only cosine terms. If $s(L-x) = -s(x)$ then the series contains only sine terms.

In the following example is how I solved the POTW question.

Example:

The POTW question asked you to evaluate:

$$\frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos 2\pi kx}{k^2} \tag{2.8}$$

for $0 \le x \le 1$.

We will evaluate the sum:

$$\frac{1}{\pi^2}\sum_{k=1}^{\infty}(-1)^k\frac{\cos 2\pi kx}{k^2} \tag{2.9}$$

for $-\frac{1}{2} \le x \le \frac{1}{2}$, and then at the end make the shift $x \mapsto x - \frac{1}{2}$.

I will express the sum via a complex contour integration. It employs the fact that the function

$$\frac{\cos 2\pi zx}{\sin \pi z} \tag{2.10}$$

has simple poles at all integer values except when $\cos 2\pi zx = 0$, as we now verify. First consider the case where $\cos 2\pi zx \neq 0$ for $z = n$,

$$\begin{aligned}
\frac{\cos 2\pi zx}{\sin \pi z} &= \frac{\cos 2\pi[n + (z-n)]x}{\sin \pi[n + (z-n)]} \\
&= \frac{\cos 2\pi nx + \cdots}{(-1)^n[\pi(z-n) - \frac{1}{3!}\pi^3(z-n)^3 + \cdots]} \\
&= (-1)^n\frac{\cos 2\pi nx + \cdots}{(z-n)\pi[1 - \frac{1}{3!}\pi^2(z-n)^2 + \cdots]} \\
&= (-1)^n\frac{\cos 2\pi nx}{(z-n)\pi} + \cdots
\end{aligned} \tag{2.11}$$

Now consider the case where $\cos 2\pi zx = 0$ for some $z = n$,

$$\begin{aligned}
\left|\lim_{z\to n}\frac{\cos 2\pi zx}{\sin \pi z}\right| &= \left|\lim_{z\to n}\frac{\frac{d}{dz}\cos 2\pi zx}{\frac{d}{dz}\sin \pi z}\right| \\
&= \left|\lim_{z\to n}\frac{-2\pi x\sin 2\pi zx}{\pi\cos \pi z}\right| < \infty
\end{aligned} \tag{2.12}$$

So when $\cos 2\pi zx$ vanishes for $z = n$, there is no pole at n. This allows us to write

$$\frac{1}{\pi^2}\sum_{k=1}^{\infty}(-1)^k\frac{\cos 2\pi kx}{k^2} = \frac{1}{2\pi^2 i}\oint_C \frac{\cos 2\pi zx}{z^2\sin\pi z}dz \tag{2.13}$$

We have

$$\begin{aligned}
2\frac{1}{\pi^2}\sum_{k=1}^{\infty}(-1)^k\frac{\cos 2\pi kx}{k^2} &= \frac{1}{\pi^2}\sum_{k=1}^{\infty}(-1)^k\frac{\cos 2\pi kx}{k^2} + \frac{1}{\pi^2}\sum_{k=-1}^{-\infty}(-1)^k\frac{\cos 2\pi kx}{k^2} \\
&= \frac{1}{2\pi^2 i}\oint_{C+C'} \frac{\cos 2\pi zx}{z^2\sin\pi z}dz
\end{aligned} \tag{2.14}$$

Integration along infinite semi-circles (see fig 1) vanishes. So that

$$\frac{1}{\pi^2}\sum_{k=1}^{\infty}(-1)^k\frac{\cos 2\pi kx}{k^2} = \frac{1}{4\pi^2 i}\oint_{C_0} \frac{\cos 2\pi zx}{z^2\sin\pi z}dz \tag{2.15}$$

We expand the integrand in powers of z about $z = 0$ and isolate the z^{-1} term. We get

$$\begin{aligned}
\frac{\cos 2\pi zx}{z^2\sin\pi z} &= \frac{1 - \frac{1}{2!}2^2\pi^2x^2z^2 + \cdots}{z^2[\pi z - \frac{1}{3!}\pi^3z^3 + \cdots]} \\
&= \frac{1 - 2\pi^2x^2z^2 + \cdots}{z^3\pi[1 - \frac{1}{6}\pi^2z^2 + \cdots]} \\
&= \frac{1 - 2\pi^2x^2z^2 + \cdots}{z^3\pi}(1 + \frac{1}{6}\pi^2z^2 + \cdots) \\
&= \cdots + \left(\frac{1}{6} - 2x^2\right)\pi\frac{1}{z} + \cdots
\end{aligned} \tag{2.16}$$

So that for $-\frac{1}{2} \leq x \leq \frac{1}{2}$,

$$\begin{aligned}
\frac{1}{\pi^2}\sum_{k=1}^{\infty}(-1)^k\frac{\cos 2\pi kx}{k^2} &= \frac{1}{4\pi^2 i}\oint_{C_0} \frac{\cos 2\pi zx}{z^2\sin\pi z}dz \\
&= \frac{1}{4\pi^2 i}(-2\pi i)\left(\frac{1}{6} - 2x^2\right)\pi \\
&= x^2 - \frac{1}{12}
\end{aligned} \tag{2.17}$$

Finally, we make the shift $x \mapsto x - \frac{1}{2}$ to obtain,

$$\frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos 2\pi kx}{k^2} = x^2 - x + \frac{1}{6} \tag{2.18}$$

for $0 \leq x \leq 1$.

$\square$

After having gone through this example, let us write down general results.

First take the case where we have a Fourier series corresponding to the interval $-\frac{1}{2} \leq x \leq \frac{1}{2}$. If

$$s(x) = \sum_{n=-\infty}^{\infty}(-1)^n h(n) e^{i2\pi nx} \tag{2.19}$$

then

$$s(x) = \sum_{n=1}^{\infty}(-1)^n f(n)\cos 2\pi nx + \sum_{n=1}^{\infty}(-1)^n g(n)\sin 2\pi nx \tag{2.20}$$

where $f(n) = h(n) + h(-n)$ and $g(n) = i(h(n) - h(-n))$. Consider applying the complex contour integration trick to the RHS of (2.20) in order to determines $s(x)$. This is viable as the functions $f(z)$ and $g(z)$ satisfy $f(-n) = f(n)$ and $g(-n) = -g(n)$. The success of this method relies on the vanishing of the integral around the infinite semi-circles, and on $f(z)$ and $g(z)$ having no branch cuts or essential singularities. Then

$$s(x) = \frac{1}{4i}\oint_{C_s}\frac{f(z)\cos 2\pi zx}{\sin \pi z}dz + \frac{1}{4i}\oint_{C_s}\frac{g(z)\sin 2\pi zx}{\sin \pi z}dz. \tag{2.21}$$

Now say we wanted to invert a Fourier series defined over the interval $0 \leq x \leq 1$. We can take

$$\begin{aligned} s(x+\frac{1}{2}) &= \sum_{n=1}^{\infty} f(n)\cos 2\pi n(x+\frac{1}{2}) + \sum_{n=1}^{\infty} g(n)\sin 2\pi n(x+\frac{1}{2}) \\ &\equiv \sum_{n=1}^{\infty}(-1)^n f(n)\cos 2\pi nx + \sum_{n=1}^{\infty}(-1)^n g(n)\sin 2\pi nx \end{aligned} \tag{2.22}$$

and perform our complex contour integration trick on this, then make the shift $x \mapsto x - \frac{1}{2}$ to obtain $s(x)$. Again, the success of this method relies on the vanishing of the integrals around the infinite semi-circles, and on $f(z)$ and $g(z)$ having no branch cuts or essential singularities. Then making the shift $x \mapsto x - \frac{1}{2}$,

$$s(x) = \frac{1}{4i} \oint_{C_s} \frac{f(z) \cos 2\pi z(x - \frac{1}{2})}{\sin \pi z} dz + \frac{1}{4i} \oint_{C_s} \frac{g(z) \sin 2\pi z(x - \frac{1}{2})}{\sin \pi z} dz \tag{2.23}$$

2.2 A Summation Theorem

We consider summations of the form $\sum_{n=1}^{\infty} (-1)^n \sin(2\pi n x) g(n)$ over the interval $-\frac{1}{2} < x < \frac{1}{2}$ where $g(-n) = -g(n)$.

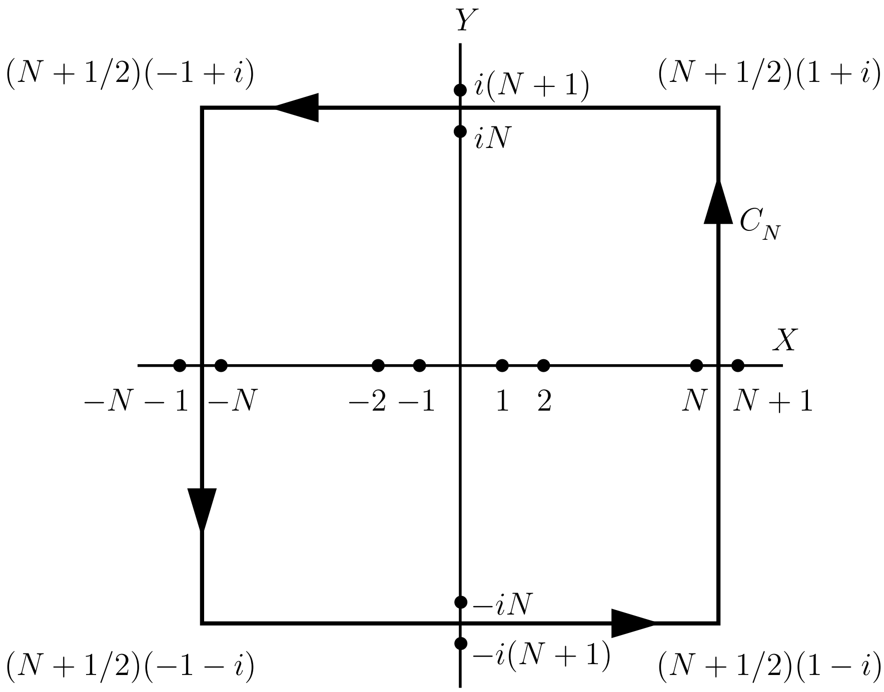

Figure 2.1: The square contour C_N.

Let $g(z)$ be analytic in $\mathbb{C}$ except for some finite set of isolated singularities. We prove that if $|g(z)| \leq \frac{M}{|z|^k}$ along the path C_N, where $k > 1$ and M are constants independent of N, then the integral around C_N vanishes. Then we briefly establish a summation theorem.

We start by showing that the value of $|\sin(2\pi x z) \csc(\pi z)|$ around the square C_N is bounded by a constant that is independent of N.

We write $z = X + iY$

Case 1: $Y > \frac{1}{2}$, we have

$$
\begin{aligned}
\left|\frac{\sin(2\pi zx)}{\sin(\pi z)}\right| &= \left|\frac{e^{i2\pi zx} - e^{-i2\pi zx}}{e^{i\pi z} - e^{-i\pi z}}\right| \\
&= \frac{|e^{i2\pi zx}| + |e^{-i2\pi zx}|}{|e^{i\pi z}| - |e^{-i\pi z}|} \\
&= \frac{|e^{i2\pi xX - xY}| + |e^{-i2\pi xX + xY}|}{|e^{i\pi X - \pi Y}| - |e^{-i\pi X + \pi Y}|} \\
&= \frac{e^{-2\pi xY} + e^{2\pi xY}}{e^{\pi Y} - e^{-\pi Y}} \\
&= \frac{e^{Y\pi(2x-1)} + e^{-Y\pi(2x+1)}}{1 - e^{-2\pi Y}} \\
&\leq \frac{2}{1 - e^{-2\pi Y}} \qquad \text{as we are taking } -\frac{1}{2} < x < \frac{1}{2} \\
&\leq \frac{2}{1 - e^{-\pi}} \qquad \text{as we are taking } Y > \frac{1}{2} \\
&=: A_1
\end{aligned} \tag{2.24}
$$

Case 2: $Y < -\frac{1}{2}$, we have

$$
\begin{aligned}
\left|\frac{\sin(2\pi zx)}{\sin(\pi z)}\right| &= \frac{|e^{i2\pi xX - 2\pi xY}| + |e^{-i2\pi xX + 2\pi xY}|}{|e^{i\pi X - \pi Y}| - |e^{-i\pi X + \pi Y}|} \\
&= \frac{e^{-2\pi xY} + e^{2\pi xY}}{e^{-\pi Y} - e^{\pi Y}} \\
&= \frac{e^{-Y\pi(2x-1)} + e^{Y\pi(2x+1)}}{1 - e^{2\pi Y}} \\
&\leq \frac{2}{1 - e^{2\pi Y}} \qquad \text{as we are taking } -\frac{1}{2} < x < \frac{1}{2} \\
&\leq \frac{2}{1 - e^{-\pi}} \qquad \text{as we are taking } Y < -\frac{1}{2} \\
&= A_1
\end{aligned} \tag{2.25}
$$

Case 3: $-\frac{1}{2} \leq Y \leq \frac{1}{2}$. We consider $z = N + \frac{1}{2} + iY$. We make repeated use of $\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta$ and $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$. In particular we will use:

$$\sin(\pi(N + \frac{1}{2} + iY)) = \cos(\pi N)\sin(\pi/2 + i\pi Y) = (-1)^N \cosh(\pi Y)$$

We have:

$$
\begin{aligned}
& \left|\frac{\sin(2\pi x(N+\frac{1}{2}+iY))}{\sin(\pi(N+\frac{1}{2}+iY))}\right| \\
= \quad & \frac{|\cos(2\pi xN)\sin(\pi x+i2\pi xY)+\sin(2\pi xN)\cos(\pi x+i2\pi xY)|}{\cosh(\pi Y)} \\
\leq \quad & \frac{|\cos(2\pi xN)\sin(\pi x+i2\pi xY)}{\cosh(\pi Y)|}+\frac{|\sin(2\pi xN)\cos(\pi x+i2\pi xY)|}{\cosh(\pi Y)} \\
\leq \quad & \frac{|\sin(\pi x+i2\pi xY)}{\cosh(\pi Y)|}+\frac{|\cos(\pi x+i2\pi xY)}{\cosh(\pi Y)|} \\
\leq \quad & \frac{|\cos(\pi x)\sin(i2\pi xY)+\sin(\pi x)\cos(i2\pi xY)|}{\cosh(\pi Y)}+\frac{|\cos(\pi x)\cos(i2\pi xY)-\sin(\pi x)\sin(i2\pi xY)|}{\cosh(\pi Y)} \\
\leq \quad & 2\frac{|\sinh(2\pi xY)|}{\cosh(\pi Y)}+2\frac{\cosh(2\pi xY)}{\cosh(\pi Y)} \\
= \quad & 2\frac{|e^{2\pi xY}-e^{-2\pi xY}|}{e^{\pi Y}+e^{-\pi Y}}+2\frac{e^{2\pi xY}+e^{-2\pi xY}}{e^{\pi Y}+e^{-\pi Y}} \\
\leq \quad & 4\frac{e^{2\pi xY}+e^{-2\pi xY}}{e^{\pi Y}+e^{-\pi Y}} \\
= \quad & 4\frac{e^{Y\pi(2x-1)}+e^{-Y\pi(2x+1)}}{1+e^{-\pi Y}} \\
< \quad & \frac{8e^{\pi}}{1+e^{-\pi/2}} \qquad \text{as we are taking } -\frac{1}{2}<x<\frac{1}{2} \text{ and } -\frac{1}{2}\leq Y\leq\frac{1}{2} \\
=: \quad & A_2
\end{aligned} \tag{2.26}
$$

So choose A such that $A > \max\{A_1, A_2\}$. Then we have $|\sin(2\pi xz)\csc(\pi z)| < A$ on C_N with an A independent of N. Then

$$
\left|\oint_{C_N}\frac{\sin(2\pi xz)}{\sin(\pi z)}g(z)dz\right| \leq \frac{\pi AM}{N^k}(8N+4)
$$

as $(8N+4)$ is the length of the curve C_N. Letting $N\to\infty$ we get that the integral vanishes.

□

This means we have proven the summation formula

$$
\begin{aligned}
\sum_{n=-\infty,n\neq 0}^{\infty}(-1)^n\sin(2\pi nx)g(n) &= -Res_{z=0}(\pi\csc(\pi z)\sin(xz)g(z)) \\
&\quad -\sum\{\text{ residues of } \pi\csc(\pi z)\sin(xz)g(z) \text{ at } g's \text{ poles for } z\neq 0\}
\end{aligned} \tag{2.27}
$$

for the interval $-\pi < x < \pi$.

As mentioned, at the end of the calculation we make the substitution $x \mapsto x - \pi$, thus obtaining

$$2\sum_{n=1}^{\infty} \sin(nx) g(n) = -Res_{z=0}(\pi \csc(\pi z) \sin(z(x-\pi)) g(z)) - \sum\{ \text{ residues of } \pi \csc(\pi z) \sin(z(x-\pi)) g(z) \text{ at}$$

for the interval $0 < x < 2\pi$ (we have used $g(-n) = -g(n)$).

2.3 Inverting Fourier series - Bernoulli polynomials

We consider particular cases of $f(z)$ and $g(z)$.

2.4 Fourier series over interval $0 \le x \le 1$

2.4.1 Fourier series $\sum_{n=1}^{\infty} \dfrac{\cos(2\pi nx)}{n^{2k}}$ over interval $0 \le x \le 1$

We invert

$$\sum_{n=1}^{\infty} (-1)^n \frac{\cos(2\pi nx)}{n^{2k}} \tag{2.28}$$

for $-\frac{1}{2} \le x \le \frac{1}{2}$. Then make the shift $x \mapsto x - \dfrac{1}{2}$.

We have

$$\begin{aligned} \frac{ze^{zx}}{\sinh \frac{z}{2}} &= \frac{2ze^{z(x+\frac{1}{2})}}{e^z - 1} \\ &= 2\sum_{n=0}^{\infty} B_n(x + \frac{1}{2}) \frac{z^n}{n!} \end{aligned} \tag{2.29}$$

Make the map $z \mapsto i2\pi z$:

$$\frac{\pi z}{\sin \pi z} e^{i2\pi z x} = \sum_{n=0}^{\infty} B_n(x+\frac{1}{2}) \frac{(i2\pi z)^n}{n!} \tag{2.30}$$

From which we obtain

$$\begin{aligned}
\frac{\pi z}{\sin \pi z} \cos 2\pi z x &= \sum_{n=0}^{\infty} \frac{[1+(-1)^n]}{2} B_n(x+\frac{1}{2}) \frac{(i2\pi z)^n}{n!} \\
&= \sum_{n=0}^{\infty} B_{2n}(x+\frac{1}{2}) \frac{(-1)^n (2\pi)^{2n}}{(2n)!} z^{2n} \qquad (2.31)
\end{aligned}$$

We have

$$\frac{1}{4i}(-2\pi i)\frac{1}{(2k)!}\frac{1}{\pi}\lim_{z\to 0}\frac{d^{2k}}{dz^{2k}}\frac{\pi z \cos 2\pi z x}{\sin \pi z} = -\frac{1}{2} B_{2k}(x+\frac{1}{2})\frac{(-1)^k (2\pi)^{2k}}{(2k)!} \tag{2.32}$$

and so

$$\sum_{n=1}^{\infty} (-1)^n \frac{\cos(2\pi n x)}{n^{2k}} = \frac{(-1)^{k+1}(2\pi)^{2k}}{2(2k)!} B_{2k}(x+\frac{1}{2}) \tag{2.33}$$

for $-\frac{1}{2} \leq x \leq \frac{1}{2}$. Then make the shift $x \mapsto x - \dfrac{1}{2}$ we finally have

$$\sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{n^{2k}} = \frac{(-1)^{k+1}(2\pi)^{2k}}{2(2k)!} B_{2k}(x) \tag{2.34}$$

for $0 \leq x \leq 1$.

Note that, for $-1/2 \leq x \leq 1/2$,

$$\sum_{n=1}^{\infty} (-1)^n \frac{\cos(2\pi n x)}{n^{2k+1}} = 0 \tag{2.35}$$

by (2.31). Making the shift $x \mapsto x - \dfrac{1}{2}$ we have

$$\sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{n^{2k+1}} = 0 \tag{2.36}$$

for $0 \leq x \leq 1$.

2.4.2 Fourier series $\sum_{n=1}^{\infty} \dfrac{\sin(2\pi nx)}{n^{2k+1}}$ over interval $0 \le x \le 1$

From (2.30) we have

$$\begin{aligned} \frac{\pi z}{\sin \pi z} \sin 2\pi z x &= \sum_{n=0}^{\infty} \frac{[1-(-1)^n]}{2} B_n(x+\frac{1}{2}) \frac{(i2\pi z)^n}{in!} \\ &= \sum_{n=0}^{\infty} B_{2n+1}(x+\frac{1}{2}) \frac{(-1)^n (2\pi)^{2n+1}}{(2n+1)!} z^{2n+1}. \end{aligned} \tag{2.37}$$

We have

$$\frac{1}{4i}(-2\pi i)\frac{1}{(2k+1)!}\frac{1}{\pi}\lim_{z\to 0}\frac{d^{2k+1}}{dz^{2k+1}}\frac{\pi z \sin 2\pi z x}{\sin \pi z} = -\frac{1}{2}B_{2k+1}(x+\frac{1}{2})\frac{(-1)^k(2\pi)^{2k+1}}{(2k+1)!} \tag{2.38}$$

and so

$$\sum_{n=1}^{\infty}(-1)^n \frac{\sin(2\pi nx)}{n^{2k}} = \frac{(-1)^{k+1}(2\pi)^{2k+1}}{2(2k+1)!} B_{2k}(x+\frac{1}{2}) \tag{2.39}$$

for $-\frac{1}{2} \le x \le \frac{1}{2}$. Then make the shift $x \mapsto x - \dfrac{1}{2}$ we finally have

$$\sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n^{2k+1}} = \frac{(-1)^{k+1}(2\pi)^{2k+1}}{2(2k+1)!} B_{2k+1}(x) \tag{2.40}$$

for $0 \le x \le 1$.

Note that, for $-1/2 \le x \le 1/2$,

$$\sum_{n=1}^{\infty}(-1)^n \frac{\sin(2\pi nx)}{n^{2k}} = 0 \tag{2.41}$$

by (2.37). Making the shift $x \mapsto x - \dfrac{1}{2}$ we have

$$\sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n^{2k}} = 0 \tag{2.42}$$

for $0 \le x \le 1$.

2.4.3 Combining the results

We have

$$\begin{aligned} B_{2k}(x) &= (-1)^{k+1}\frac{(2k)!}{(2\pi)^{2k}}\sum_{n\neq 0}^{\infty}\frac{\cos(2\pi nx)}{n^{2k}} \\ B_{2k+1}(x) &= (-1)^{k+1}\frac{(2k+1)!}{(2\pi)^{2k+1}}\sum_{n\neq 0}^{\infty}\frac{\sin(2\pi nx)}{n^{2k+1}}. \end{aligned} \tag{2.43}$$

These, with the use of (2.36) and (2.42), can be combined into a single expression:

$$\begin{aligned} B_k(x) &= -\frac{k!}{(2\pi i)^k}\sum_{n\neq 0}^{\infty}\frac{\cos(2\pi nx)}{n^k} - \frac{k!}{(2\pi i)^k}\sum_{n\neq 0}^{\infty}\frac{i\sin(2\pi nx)}{n^k} \\ &= -\frac{k!}{(2\pi i)^k}\sum_{n\neq 0}^{\infty}\frac{e^{2\pi inx}}{n^k}. \end{aligned} \tag{2.44}$$

Fourier series for monomials

Using the inversion formula, (C.14),

$$x^m = -\frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k(x) \tag{2.45}$$

we have

$$x^m = -\frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}\frac{k!}{(2\pi i)^k}\sum_{n\neq 0}^{\infty}\frac{e^{2\pi inx}}{n^k}. \tag{2.46}$$

2.4.4 Fourier series inversion - arbitrary polynomial

Say we were given the Fourier series:

$$\sum_{k=0}^{l} d_k \sum_{n\neq 0}^{\infty}\frac{e^{2\pi inx}}{n^k}. \tag{2.47}$$

Obviously, the function this Fourier series represents will be a polynomial. Substituting in (2.46) into $\sum_{m=0}^{l} a_m x^m$, we have

$$\begin{aligned}\sum_{m=0}^{l} a_m x^m &= -\sum_{m=0}^{l} \frac{a_m}{m+1} \sum_{k=1}^{m} \binom{m+1}{k} \frac{k!}{(2\pi i)^k} \sum_{n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{n^k} \\ &= -\sum_{k=1}^{l} \sum_{m=k}^{l} \frac{a_m}{m+1} \binom{m+1}{k} \frac{k!}{(2\pi i)^k} \sum_{n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{n^k} \end{aligned} \tag{2.48}$$

Comparing the RHS of this with (2.47), we have

$$d_k = -\sum_{m=k}^{l} \frac{1}{m+1} \binom{m+1}{k} \frac{k!}{(2\pi i)^k} a_m. \tag{2.49}$$

We define

$$B_{k,m} = -\frac{1}{m+1} \binom{m+1}{k} \frac{k!}{(2\pi i)^k}. \tag{2.50}$$

Then (2.49) reads

$$\begin{pmatrix} B_{0,0} & B_{0,1} & B_{0,2} & \cdots & B_{0,l} \\ 0 & B_{1,1} & B_{1,2} & \cdots & B_{1,l} \\ 0 & 0 & B_{2,2} & \cdots & B_{2,l} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & B_{l,l} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_l \end{pmatrix} = \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_l \end{pmatrix} \tag{2.51}$$

As the equation we is in triangular form, we easily solve using back-substitution.

Alternatively, we can write down explicit expressions for the a_i's using Cramer's rule:

$$a_i = \frac{\begin{vmatrix} B_{0,0} & B_{0,1} & B_{0,2} & \cdots & d_0 & \cdots & B_{0,l} \\ 0 & B_{1,1} & B_{1,2} & \cdots & d_1 & \cdots & B_{1,l} \\ 0 & 0 & B_{2,2} & \cdots & d_2 & \cdots & B_{2,l} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & d_l & \cdots & B_{l,l} \end{vmatrix}}{\prod_{j=0}^{l} B_{j,j}} \tag{2.52}$$

where the column of d_i's are in the ith column.

2.5 Fourier series over interval $0 \leq x \leq 1$ and Euler polynomials

2.5.1 Fourier series $\sum_{n=0}^{\infty} \dfrac{\cos(\pi(2n+1)x)}{(2n+1)^{2k}}$ over interval $0 \leq x \leq 1$

We wish to invert the Fourier series

$$\sum_{n=0}^{\infty} \frac{\cos(\pi(2n+1)x)}{(2n+1)^{2k}} \tag{2.53}$$

over interval $0 \leq x \leq 1$. So we invert

$$\sum_{n=0}^{\infty} (-1)^{n-1} \frac{\sin(\pi(2n+1)x)}{(2n+1)^{2k}} \tag{2.54}$$

for $-\frac{1}{2} \leq x \leq \frac{1}{2}$. Then make the shift $x \mapsto x - \dfrac{1}{2}$.

We write

$$\sum_{n=0}^{\infty} (-1)^{n-1} \frac{\sin(\pi(2n+1)x)}{(2n+1)^{2k}} = \sum_{n=1}^{\infty} (-1)^n \sin\frac{\pi n}{2} \frac{\sin(\pi n x)}{n^{2k}} \tag{2.55}$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} (-1)^n \sin\frac{\pi n}{2} \frac{\sin(\pi n x)}{n^{2k}} &= \frac{1}{4i} \oint \frac{\sin\frac{\pi z}{2} \sin(\pi z x)}{z^{2k} \sin \pi z} dz \\ &= \frac{1}{8i} \oint \frac{\sin(\pi z x)}{z^{2k} \cos\frac{\pi z}{2}} dz \end{aligned} \tag{2.56}$$

We have

$$\begin{aligned}\frac{e^{zx}}{\cosh\frac{z}{2}} &= \frac{2e^{z(x+\frac{1}{2})}}{e^z+1} \\ &= \sum_{n=0}^{\infty} E_n(x+\frac{1}{2})\frac{z^n}{n!} \qquad (2.57)\end{aligned}$$

Make the map $z \mapsto i\pi z$:

$$\frac{1}{\cos\frac{\pi z}{2}} e^{i\pi z x} = \sum_{n=0}^{\infty} E_n(x+\frac{1}{2})\frac{(i\pi z)^n}{n!} \qquad (2.58)$$

From which we obtain

$$\begin{aligned}\frac{1}{z\cos\frac{\pi z}{2}}\sin\pi z x &= \sum_{n=0}^{\infty}\frac{[1-(-1)^n]}{2}E_n(x+\frac{1}{2})\frac{(i\pi z)^n}{izn!} \\ &= \sum_{n=0}^{\infty} E_{2n+1}(x+\frac{1}{2})\frac{(-1)^n\pi^{2n+1}}{(2n+1)!}z^{2n} \qquad (2.59)\end{aligned}$$

We have

$$\frac{1}{8i}(-2\pi i)\frac{1}{(2k-1)!}\lim_{z\to 0}\frac{d^{2k-2}}{dz^{2k-2}}\frac{\sin\pi z x}{z\cos\frac{\pi z}{2}} = \frac{1}{4}E_{2k-1}(x+\frac{1}{2})\frac{(-1)^k\pi^{2k}}{(2k-1)!} \qquad (2.60)$$

and so

$$\sum_{n=0}^{\infty}(-1)^{n-1}\frac{\sin\pi(2n+1)x}{(2n+1)^{2k}} = \frac{(-1)^k\pi^{2k}}{4(2k-1)!}E_{2k-1}(x+\frac{1}{2}) \qquad (2.61)$$

for $-\frac{1}{2} \le x \le \frac{1}{2}$. Then make the shift $x \mapsto x - \dfrac{1}{2}$ we finally have

$$\sum_{n=0}^{\infty}\frac{\cos\pi(2n+1)x}{(2n+1)^{2k+1}} = \frac{(-1)^k\pi^{2k}}{4(2k-1)!}E_{2k-1}(x) \qquad (2.62)$$

for $0 \le x \le 1$.

2.5.2 Fourier series $\sum_{n=0}^{\infty} \dfrac{\sin(\pi(2n+1)x)}{(2n+1)^{2k+1}}$ over interval $0 \le x \le 1$

We wish to invert the Fourier series

$$\sum_{n=0}^{\infty} \frac{\sin(\pi(2n+1)x)}{(2n+1)^{2k+1}} \tag{2.63}$$

over interval $0 \le x \le 1$. So we invert

$$\sum_{n=0}^{\infty} (-1)^n \frac{\cos(\pi(2n+1)x)}{(2n+1)^{2k+1}} \tag{2.64}$$

for $-\frac{1}{2} \le x \le \frac{1}{2}$. Then make the shift $x \mapsto x - \dfrac{1}{2}$.

We write

$$\sum_{n=0}^{\infty} (-1)^n \frac{\cos(\pi(2n+1)x)}{(2n+1)^{2k+1}} = -\sum_{n=1}^{\infty} (-1)^n \sin\frac{\pi n}{2} \frac{\cos(\pi n x)}{n^{2k+1}} \tag{2.65}$$

and

$$\begin{aligned} -\sum_{n=1}^{\infty} (-1)^n \sin\frac{\pi n}{2} \frac{\cos(\pi n x)}{n^{2k+1}} &= -\frac{1}{4i} \oint \frac{\sin\dfrac{\pi z}{2}\cos(\pi z x)}{z^{2k+1}\sin \pi z} dz \\ &= -\frac{1}{8i} \oint \frac{\cos(\pi z x)}{z^{2k+1}\cos\dfrac{\pi z}{2}} dz \end{aligned} \tag{2.66}$$

From (2.58) we have

$$\begin{aligned} \frac{1}{\cos\dfrac{\pi z}{2}} \cos \pi z x &= \sum_{n=0}^{\infty} \frac{[1+(-1)^n]}{2} E_n(x+\frac{1}{2}) \frac{(i\pi z)^n}{n!} \\ &= \sum_{n=0}^{\infty} E_{2n}(x+\frac{1}{2}) \frac{(-1)^n \pi^{2n}}{(2n)!} z^{2n} \end{aligned} \tag{2.67}$$

We have

$$-\frac{1}{8i}(-2\pi i)\frac{1}{(2k)!}\lim_{z\to 0}\frac{d^{2k}}{dz^{2k}}\frac{\cos\pi zx}{\cos\dfrac{\pi z}{2}} = \frac{1}{4}E_{2k}(x+\frac{1}{2})\frac{(-1)^k\pi^{2k}}{(2k)!} \tag{2.68}$$

and so

$$\sum_{n=0}^{\infty}(-1)^n\frac{\cos(\pi nx)}{(2n+1)^{2k+1}} = \frac{(-1)^k\pi^{2k+1}}{4(2k)!}E_{2k}(x+\frac{1}{2}) \tag{2.69}$$

for $-\frac{1}{2}\leq x\leq\frac{1}{2}$. Then make the shift $x\mapsto x-\dfrac{1}{2}$ we finally have

$$\sum_{n=0}^{\infty}\frac{\sin\pi(2n+1)x}{(2n+1)^{2k+1}} = \frac{(-1)^k\pi^{2k+1}}{4(2k)!}E_{2k}(x) \tag{2.70}$$

for $0\leq x\leq 1$.

2.6 Inverting other Fourier series

Example:

Invert

$$\frac{\sinh\pi a}{\pi}\left[\frac{1}{a}+2\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\left(a\cos nx - n\sin nx\right)\right] \tag{2.71}$$

for $-\pi\leq x\leq\pi$.

Proof:

We invert

$$s_c(x) = \frac{2\sinh\pi a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}a\cos nx \tag{2.72}$$

first.

$$\begin{aligned}
s_c(x) &= \frac{2\sinh \pi a}{\pi}\frac{1}{4i}\oint_{C_s}\frac{a\cos zx}{[a^2+z^2]\sin \pi z}dz \\
&= \frac{2\sinh \pi a}{\pi}\frac{1}{4i}(-2\pi i)\left[\frac{1}{a\pi}+\frac{a\cos iax}{2ia\sin i\pi a}+\frac{a\cos -iax}{-2ia\sin -i\pi a}\right] \\
&= -\frac{\sinh \pi a}{a\pi}+\cosh ax. \qquad (2.73)
\end{aligned}$$

Next

$$s_s(x) := -\frac{2\sinh \pi a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}2\pi n\sin nx \qquad (2.74)$$

$$\begin{aligned}
s_s(x) &= -\frac{2\sinh \pi a}{\pi}\frac{1}{4i}\oint_{C_s}\frac{z\sin zx}{[a^2+z^2]\sin \pi z}dz \\
&= -\frac{2\sinh \pi a}{\pi}\frac{1}{4i}(-2\pi i)\left[\frac{ia\sin iax}{2ia\sin i\pi a}+\frac{-ia\sin -iax}{-2ia\sin -i\pi a}\right] \\
&= \sinh ax. \qquad (2.75)
\end{aligned}$$

Finally,

$$\begin{aligned}
s(x) &= \frac{\sinh \pi a}{\pi a}+s_c(x)+s_s(x) \\
&= \frac{\sinh \pi a}{\pi a}+\left[-\frac{\sinh \pi a}{a\pi}+\cosh ax\right]+\sinh ax \\
&= e^{ax}. \qquad (2.76)
\end{aligned}$$

□

Chapter 3

Summary of results

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^2}{90} \tag{3.1}$$

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = (-1)^{k+1}(2\pi)^{2k}\frac{B_{2k}}{2(2k)!} \tag{3.2}$$

where B_{2k} are Bernoulli numbers.

The first few examples are

$$\begin{aligned}
\zeta(2) &= \sum_{n=1}^{\infty} \frac{1}{n^2} = (-1)^2(2\pi)^2\frac{B_2}{2(2)!} = \pi^2 B_2 = \frac{\pi^2}{6} = \frac{\pi^2}{6} \\
\zeta(4) &= \sum_{n=1}^{\infty} \frac{1}{n^4} = (-1)^3(2\pi)^4\frac{B_4}{2(4)!} = -\frac{2^3\pi^4}{4!}\pi^2 B_4 = \frac{\pi^4}{90} \\
\zeta(6) &= \sum_{n=1}^{\infty} \frac{1}{n^6} = (-1)^4(2\pi)^6\frac{B_6}{2(6)!} = \frac{2^6\pi^6}{2\cdot 6!}B_6 = \frac{\pi^2}{945}
\end{aligned} \tag{3.3}$$

where B_k are the Bernoulli numbers.

The Dirichlet eta function is related to the zeta function:

$$\eta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} - 2\sum_{n=1}^{\infty} \frac{1}{(2n)^k} = \frac{2^{k-1}-1}{2^{k-1}}\zeta(k) \tag{3.4}$$

We have an expression for $\csc(x)$ in terms of the Bernoulli numbers:

$$x \csc x = \sum_{n=0}^{\infty} \frac{2(2^{2n-1}-1)(-1)^{n+1} B_{2n}}{(2n)!} x^{2n} \tag{3.5}$$

Liebniz formula for π

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}. \tag{3.6}$$

Sum over odd integers - alternating case

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^{2k+1}} = \frac{(-1)^k (\pi)^{2k+1} E_{2k}}{(2k)! 4^{k+1}} \tag{3.7}$$

numbers. where E_n are Euler numbers. The first three Euler numbers are $E_0 = 1$, $E_2 = -1$, $E_4 = 5$. So, the first few examples are

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)} &= \frac{\pi}{4} E_0 = \frac{\pi}{4} \\
\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^3} &= (-1)\pi^3 \frac{E_2}{(2!)4^2} = \frac{\pi^3}{32} \\
\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^5} &= (-1)^2 \pi^5 \frac{E_4}{(4)!4^3} = \frac{\pi^5}{1536} E_4 = \frac{5\pi^5}{1536}
\end{aligned} \tag{3.8}$$

Partial fraction expansions for trigonomteric functions

$$\pi \cot \pi x = \frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{1}{x^2 - n^2} \tag{3.9}$$

$$\pi \csc \pi x = \sum_{n=1}^{\infty} \frac{(-1)^n}{x+n} = \frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{(-1)^n}{x^2 - n^2} \tag{3.10}$$

$$\pi^2 \csc^2 \pi x = \sum_{n=1}^{\infty} \frac{1}{(x+n)^2} \tag{3.11}$$

$$\pi \sec \pi x = \sum_{n=0}^{\infty} (-1)^n \frac{2n+1}{(n+\frac{1}{2})^2 - x^2} \tag{3.12}$$

$$\pi \tan \pi x = 2x \sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2})^2 - x^2} \tag{3.13}$$

Infinite products

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right) \tag{3.14}$$

$$\cos x = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{(n-1/2)^2\pi^2}\right) \tag{3.15}$$

$$\sin x \sinh x = x^2 \prod_{n=1}^{\infty} \left(1 - \frac{x^4}{n^4\pi^4}\right). \tag{3.16}$$

$$i \left[\prod_{q=0}^{n-1} \sin[e^{\frac{i2\pi q}{2k}}|x|]\right]^{1/2} = x^k \prod_{n=1}^{\infty} \left(1 - \frac{x^{2k}}{n^{2k}\pi^{2k}}\right). \tag{3.17}$$

$$\prod_{n=-\infty}^{\infty} \frac{n^k - x^k}{n^k - y^k} = \prod_{n=0}^{k-1} \frac{\sin[\pi x e^{\frac{i2n\pi}{k}}]}{\sin[\pi y e^{\frac{i2n\pi}{k}}]}. \tag{3.18}$$

$$\prod_{n=1}^{\infty} \left(1 - \frac{x}{(n^2+y)^2}\right) = \frac{y}{\sqrt{x-y^2}} \frac{\sinh[\pi\sqrt{\sqrt{x}+y}] \sin[\pi\sqrt{\sqrt{x}-y}]}{\sinh^2[\pi\sqrt{y}]}. \tag{3.19}$$

Fourier series for Bernoulli polynomials

The results

$$\begin{aligned} B_{2k}(x) &= (-1)^{k+1} \frac{(2k)!}{(2\pi)^{2k}} \sum_{n\neq 0}^{\infty} \frac{\cos(2\pi n x)}{n^{2k}} \\ B_{2k+1}(x) &= (-1)^{k+1} \frac{(2k+1)!}{(2\pi)^{2k+1}} \sum_{n\neq 0}^{\infty} \frac{\sin(2\pi n x)}{n^{2k+1}} \end{aligned} \tag{3.20}$$

Combined into a single expression:

$$
\begin{aligned}
B_k(x) &= -\frac{k!}{(2\pi i)^k}\sum_{n\neq 0}^{\infty}\frac{\cos(2\pi n x)}{n^k} - \frac{k!}{(2\pi i)^k}\sum_{n\neq 0}^{\infty}\frac{i\sin(2\pi n x)}{n^k} \\
&= -\frac{k!}{(2\pi i)^k}\sum_{n\neq 0}^{\infty}\frac{e^{2\pi i n x}}{n^k}
\end{aligned} \tag{3.21}
$$

Fourier series for Euler polynomials

$$
\sum_{n=1}^{\infty}\frac{\cos\pi(2n+1)x}{(2n+1)^{2k+1}} = \frac{(-1)^k\pi^{2k}}{4(2k-1)!}E_{2k-1}(x) \tag{3.22}
$$

and

$$
\sum_{n=0}^{\infty}\frac{\sin\pi(2n+1)x}{(2n+1)^{2k+1}} = \frac{(-1)^k\pi^{2k+1}}{4(2k)!}E_{2k}(x) \tag{3.23}
$$

for $0 \le x \le 1$.

Fourier series for monomials

Using the inversion

$$
x^m = -\frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k(x) \tag{3.24}
$$

we have

$$
x^m = -\frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}\frac{k!}{(2\pi i)^k}\sum_{n\neq 0}^{\infty}\frac{e^{2\pi i n x}}{n^k}. \tag{3.25}
$$

Appendix A

Fucntions in complex plane

$$\begin{aligned} \sin z &= \sin x \cosh y + i \cos x \sinh y \\ \cos z &= \cos x \cosh y - i \sin x \sinh y \end{aligned} \tag{A.1}$$

Appendix B

Bernoulli numbers and Euler numbers

Appendix C

Bernoulli polynomials and Euler polynomials

C.1 Bernoulli polynomials

The generating function for the Bernoulli polynomials is

$$\frac{te^{tx}}{e^t - 1} = \sum_n^{\infty} B_n(x) \frac{t^n}{n!} \tag{C.1}$$

Explicit expressions for low degrees:

$$\begin{aligned}
B_0(x) &= 1 \\
B_1(x) &= x - \frac{1}{2} \\
B_2(x) &= x^2 - x - \frac{1}{6} \\
B_3(x) &= x^3 - \frac{3}{2}x^2 - \frac{1}{2}x \\
B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30}.
\end{aligned} \tag{C.2}$$

Derivatives

We have

$$\sum_{n=0}^{\infty} B'_n(x) \frac{t^n}{n!} = \frac{t^2 e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} n B_{n-1}(x) \frac{t^n}{n!} \tag{C.3}$$

which implies

$$B_n'(x) = nB_{n-1}(x). \tag{C.4}$$

Symmetries:

$$\begin{aligned} B_n(1-x) &= (-1)^n B_n(x), \quad n \geq 0 & \text{(C.5)} \\ (-1)^n B_n(-x) &= B_n(x) - nx^{n-1} & \text{(C.6)} \end{aligned}$$

proof:

We have

$$\begin{aligned} \sum_{n=0}^{\infty} B_n(1-x)\frac{t^n}{n!} &= \frac{te^{t(1-x)}}{e^t-1} \\ &= \frac{te^t e^{t(-x)}}{e^t-1} \\ &= \frac{(-t)e^{(-t)x}}{e^{-t}-1} \\ &= \sum_{n=0}^{\infty}(-1)^n B_n(x)\frac{t^n}{n!}. & \text{(C.7)} \end{aligned}$$

Equating coefficient of eams power of t gives (C.5).

We have

$$\begin{aligned} \sum_{n=0}^{\infty}(-1)^n B_n(-x)\frac{t^n}{n!} - \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} &= \frac{-te^{tx}}{e^{-t}-1} - \frac{te^{tx}}{e^t-1} \\ &= te^{tx}\left[\frac{e^t}{e^t-1} - \frac{1}{e^t-1}\right] \\ &= te^{tx} \\ &= \sum_{n=0}^{\infty} x^n \frac{t^{n+1}}{n!} \\ &= \sum_{n=0}^{\infty} nx^n \frac{t^n}{n!}. & \text{(C.8)} \end{aligned}$$

Equating coefficient of eams power of t gives (C.6).

$\square$

Representation by a differential operator

The Bernoulli polynomials are also given by

$$B_n(x) = \frac{D}{e^D - 1} x^n \tag{C.9}$$

where $D = d/dx$ is differentiation with respect to x and the fraction is expanded as a formal power series. This follows from:

$$\sum_{n=0}^{\infty} \frac{D}{e^D - 1} x^n \frac{t^n}{n!} = \frac{D}{e^D - 1} e^{tx} = \frac{te^{tx}}{e^t - 1}. \tag{C.10}$$

Representation by an integral operator

The Bernoulli polynomials are also the unique polynomials determined by

$$\int_x^{x+1} B_n(u) du = x^n. \tag{C.11}$$

The integral transform

$$(Tf)(x) = \int_x^{x+1} f(u) du, \tag{C.12}$$

on polynomials f. If $F(x) = \int^x f(u)du$, then

$$\begin{aligned} (Tf)(x) &= \int_x^{x+1} \frac{d}{du} F(u) du \\ &= F(x+1) - F(x) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n}{dx^n} F(x) - F(x) \\ &= \sum_{n=0}^{\infty} \frac{D^n}{(n+1)!} f(x) \\ &= \frac{e^D - 1}{D} f(x) \end{aligned} \tag{C.13}$$

Inversion

$$
\begin{aligned}
x^n &= \frac{e^D - 1}{D} B_n(x) \\
&= \sum_{m=0}^{\infty} \frac{D^m}{(m+1)!} B_n(x) \\
&= \sum_{m=0}^{n} \frac{n!}{(m+1)!(n-m)!} B_{n-m}(x) \\
&= \frac{1}{n+1} \sum_{k=0}^{n} \frac{(n+1)!}{(n+1-k)!k!} B_k(x) \\
&= \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} B_k(x). \qquad \text{(C.14)}
\end{aligned}
$$

C.2 Euler polynomials

The generating function for the Euler polynomials is

$$
\frac{2e^{tx}}{e^t + 1} = \sum_{n}^{\infty} E_n(x) \frac{t^n}{n!} \qquad \text{(C.15)}
$$

Explicit expressions for low degrees:

$$
\begin{aligned}
E_0(x) &= 1 \\
E_1(x) &= x - \frac{1}{2} \\
E_2(x) &= x^2 - x \\
E_3(x) &= x^3 - \frac{3}{2}x^3 + \frac{1}{4} \\
E_4(x) &= x^4 - 2x^3 + x. \qquad \text{(C.16)}
\end{aligned}
$$

Symmetries:

$$
\begin{aligned}
E_n(1-x) &= (-1)^n E_n(x), \quad n \geq 0 \qquad \text{(C.17)} \\
(-1)^n E_n(-x) &= -E_n(x) + 2x^n. \qquad \text{(C.18)}
\end{aligned}
$$

proof:

We have

$$
\begin{aligned}
\sum_{n=0}^{\infty} E_n(1-x)\frac{t^n}{n!} &= \frac{2e^{t(1-x)}}{e^t+1} \\
&= \frac{2e^t e^{t(-x)}}{e^t+1} \\
&= \frac{2e^{(-t)x}}{e^{-t}+1} \\
&= \sum_{n=0}^{\infty}(-1)^n E_n(x)\frac{t^n}{n!}. \qquad \text{(C.19)}
\end{aligned}
$$

Equating coefficients of the same power of t gives (C.17).

We have

$$
\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n E_n(-x)\frac{t^n}{n!} + \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} &= \frac{2e^{tx}}{e^{-t}+1} + \frac{2e^{tx}}{e^t+1} \\
&= 2e^{tx}\left[\frac{e^t}{e^t+1} + \frac{1}{e^t+1}\right] \\
&= 2e^{tx} \\
&= \sum_{n=0}^{\infty} x^n \frac{t^n}{n!}. \qquad \text{(C.20)}
\end{aligned}
$$

Equating coefficient of the same power of t gives (C.18).

□