$$A = \sum_{k=1}^{n} a_k e_k, \quad \|A\| = (A\cdot A)^{1/2},$$

$$\|A+B\|^2 \leq (\|A\|+\|B\|)^2, \quad (A+B)\cdot(A+B) = A\cdot A + 2A\cdot B + B\cdot B$$

$$= \|A\|^2 + 2A\cdot B + \|B\|^2, \quad (\|A\|+\|B\|)^2 = \|A\|^2 + 2\|A\|\,\|B\| + \|B\|^2$$

$$\therefore \quad \boxed{A\cdot B \leq \|A\|\,\|B\|}, \quad A\cdot B \leq |A\cdot B| \leq \|A\|\,\|B\|$$

TE results also from Cauchy-Schwarz

Cauchy-Schwarz

$$|A\cdot B| \leq \|A\|\,\|B\|$$

$$(A\cdot B)^2 \leq \|A\|^2\|B\|^2$$

$$\downarrow$$

$$|A\cdot B| \leq \|A\|\,\|B\|$$