

Integral Proofs and MVT Practice

Practice Problems

Don't use the Internet, use your text book to help.

1. Note that if f is positive and continuous on $[a, b]$, then there exists at least one $c \in [a, b]$ such that

$$\int_a^b f(t) dt = f(c)(b - a).$$

Prove that this conclusion continues to hold for any differentiable function f by applying the mean value theorem on the interval $[a, b]$ to the function $A(x)$ defined by

$$A(x) = \int_a^x f(t) dt.$$

2. Given that f has a continuous derivative, does

$$\frac{d}{dx} \left[\int_a^x f(t) dt \right] = \int_a^x \frac{d}{dt} [f(t)] dt ?$$

Explain why or why not. (Note that the above statement says that, if $f \in \mathcal{C}^1$, then we can change the order of the derivative and integral, i.e., we can bring the derivative inside the integral.)

3. Let $F(x) = \int_1^{x^2} \cos t dt$. Note that $F(x) = (f \circ g)(x)$ where $g(x) = x^2$ and $f(x) = \int_1^x \cos t dt$.

Use the chain rule to compute $\frac{d}{dx} F(x)$.

4. Given that f is a continuous function, let

$$F(x) = \int_0^x x f(t) dt.$$

Find $F'(x)$. *Hint:* The answer is not $xf(x)$.

5. Evaluate without doing any algebraic computations. (Consider any relevant symmetry or geometry related to the function being integrated or the region over which we are integrating.)

(a) $\int_{-1}^1 x^3 \sqrt{1 - x^2} dx$

(b) $\int_{-1}^1 (x^5 + 3) \sqrt{1 - x^2} dx$

6. Suppose u and v are differentiable and f is continuous. Show that

$$\frac{d}{dx} \left(\int_{u(x)}^{v(x)} f(t) dt \right) = f(v(x))v'(x) - f(u(x))u'(x)$$

Hint: Break the integral up into $\int_{u(x)}^c f(t) dt + \int_c^{v(x)} f(t) dt$.