First we define known quantities

In[49]:=
$$\Delta := r^2 - 2\,M\,r + a^2$$
$$\Sigma := r^2 + a^2\,\mathrm{Cos}[\theta]^2$$
$$J := C\,\mathrm{DiracDelta}[r - r0]\,\mathrm{DiracDelta}\left[\theta - \frac{\pi}{2}\right]$$
$$\mathrm{Jm\dagger} := \left(\sqrt{2}\,\left(r - i\,a\,\mathrm{Cos}[\theta]\right)\right)^{-1}\,i\left(r^2 + a^2\right)\mathrm{Sin}[\theta]\,J$$
$$\mathrm{Jn} := -\frac{a\,\Delta}{\Sigma}\,\mathrm{Sin}[\theta]^2\,J$$

Now for J2, we can ignore ∂_φ since Jm and Jn are independent of it

$$J_2 = \frac{-\Delta}{2\sqrt{2}\,\Sigma\,(r - ia\cos\theta)^2}\left[\sqrt{2}\left(\frac{\partial}{\partial r} - \frac{a}{\Delta}\frac{\partial}{\partial\varphi} + \frac{1}{r - ia\cos\theta}\right)(r - ia\cos\theta)^2 J_{\overline{m}} + 2\left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi} + \frac{ia\sin\theta}{r - ia\cos\theta}\right)\frac{\Sigma\,(r - ia\cos\theta)}{\Delta}J_n\right] \qquad (2.12)$$

In[54]:=
$$\mathrm{J2} := \frac{-\Delta}{2\,\sqrt{2}\,\Sigma\left(r - i\,a\,\mathrm{Cos}[\theta]\right)^2}\left(\sqrt{2}\left(\partial_r \# + \frac{1}{r - i\,a\,\mathrm{Cos}[\theta]}\,\#\right)\&@\left(\left(r - i\,a\,\mathrm{Cos}[\theta]\right)^2\,\mathrm{Jm\dagger}\right) + 2\left(\partial_\theta \# + \frac{i\,a\,\mathrm{Sin}[\theta]}{r - i\,a\,\mathrm{Cos}[\theta]}\,\#\right)\&@\left(\frac{\Sigma\left(r - i\,a\,\mathrm{Cos}[\theta]\right)}{\Delta}\,\mathrm{Jn}\right)\right)$$

Finally for the integration we need to first look at the definition of the SpinWeighted spherical harmonics.

Since all SpinWeighted spherical harmonics, here denoted ${}_sY_{lm}(\theta,\phi)$= Y[s,l,θ,ϕ] behave in the ϕ argument as Exp[i m ϕ] and since all other variables are independent of ϕ in the equation above We have

$\int$Y[s,l,m,θ,ϕ]= 2 π DiracDelta[m,0] Y[s,l,m,θ,0]. Then we can integrate over θ.

the source term is given by

$$^2J_{lm}(r) = \int_0^{2\pi}\int_0^{\pi}\frac{(r - ia\cos\theta)^2}{(r_+ - r_-)^2}\;\Sigma\,J_2\;{}_{-1}\overline{Y}_{lm}\,\sin\theta\,d\theta\,d\varphi$$

In[55]:=

We can split the integrand into two parts depending if we have DiracDelta$\left[\theta - \frac{\pi}{2}\right]$ or the derivative DiracDelta'$\left[\theta - \frac{\pi}{2}\right]$.

and we will use the standard properties

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$$

$$\int f(x)\,\delta^{(n)}(x)\,dx \equiv -\int \frac{\partial f}{\partial x}\,\delta^{(n-1)}(x)\,dx.$$

First part

In[56]:=

$$\frac{(r - i\, a \operatorname{Cos}[\theta])^2}{(rp - rm)^2}\, \Sigma\, J2\, \text{Conjugate}[Y[-1, l, 0, \theta, 0]];$$

$$\%\ /.\ \text{DiracDelta}[-\pi + 2\,\theta] \to 1\ /.\ \text{DiracDelta}'[-\pi + 2\,\theta] \to 0;$$

$$\text{FirstPart} = \%\ /.\ \theta \to \frac{\pi}{2}$$

Out[58]=

$$-\frac{1}{2\sqrt{2}\,(-rm + rp)^{\text{`}}}\left(a^{\text{`}} - 2\,M\,r + r^{\text{`}}\right) \text{Conjugate}\left[Y\left[-1, l, 0, \frac{\pi}{2}, 0\right]\right]$$
$$\left(-8\,i\,a^{\text{`}}\,C\,\text{DiracDelta}[r - r0] + \sqrt{2}\left(2\,i\,\sqrt{2}\,C\,r^{\text{`}}\,\text{DiracDelta}[r - r0] + \right.\right.$$
$$\left.\left. 2\,i\,\sqrt{2}\,C\left(a^{\text{`}} + r^{\text{`}}\right)\text{DiracDelta}[r - r0] + i\,\sqrt{2}\,C\,r\left(a^{\text{`}} + r^{\text{`}}\right)\text{DiracDelta}'[r - r0]\right)\right)$$

Second part

In[59]:=

$$\frac{(r - i\, a \operatorname{Cos}[\theta])^2}{(rp - rm)^2}\, \Sigma\, J2\, \text{Conjugate}[Y[-1, l, 0, \theta, 0]];$$

$$\%\ /.\ \text{DiracDelta}[-\pi + 2\,\theta] \to 0\ /.\ \text{DiracDelta}'[-\pi + 2\,\theta] \to 1;$$

$$\text{SecondPart} = -D[\%, \theta]\ /.\ \theta \to \frac{\pi}{2}\ /.\ \text{Conjugate}'\left[Y\left[-1, l, 0, \frac{\pi}{2}, 0\right]\right] \to 1$$

Out[61]=

$$-\frac{2\,i\,\sqrt{2}\,a^{\text{`}}\,C\left(a^{\text{`}} - 2\,M\,r + r^{\text{`}}\right)\text{Conjugate}\left[Y\left[-1, l, 0, \frac{\pi}{\text{`}}, 0\right]\right]\text{DiracDelta}[r - r0]}{(-rm + rp)^{\text{`}}} -$$
$$\frac{2\sqrt{2}\,a\,C\,r\left(a^{\text{`}} - 2\,M\,r + r^{\text{`}}\right)\text{DiracDelta}[r - r0]\,Y^{(\text{.И.И.И.И.})}\left[-1, l, 0, \frac{\pi}{\text{`}}, 0\right]}{(-rm + rp)^{\text{`}}}$$

We get the solution

In[62]:=
```
solution = FirstPart + SecondPart // Simplify
```

Out[62]=

$$-\frac{2\,i\,\sqrt{2}\;a^{\text{`}}\;C\left(a^{\text{`}}-2\,M\,r+r^{\text{`}}\right)\texttt{Conjugate}\left[Y\left[-1,l,0,\tfrac{\pi}{\text{`}},0\right]\right]\texttt{DiracDelta}[r-r0]}{(-rm+rp)^{\text{`}}}-$$

$$\frac{1}{2\sqrt{2}\,(-rm+rp)^{\text{`}}}\left(a^{\text{`}}-2\,M\,r+r^{\text{`}}\right)\texttt{Conjugate}\left[Y\left[-1,l,0,\frac{\pi}{2},0\right]\right]$$

$$\Big(-8\,i\,a^{\text{`}}\,C\,\texttt{DiracDelta}[r-r0]+\sqrt{2}\,\Big(2\,i\,\sqrt{2}\;C\,r^{\text{`}}\;\texttt{DiracDelta}[r-r0]+$$

$$2\,i\,\sqrt{2}\;C\left(a^{\text{`}}+r^{\text{`}}\right)\texttt{DiracDelta}[r-r0]+i\,\sqrt{2}\;C\,r\left(a^{\text{`}}+r^{\text{`}}\right)\texttt{DiracDelta}'[r-r0]\Big)\Big)-$$

$$\frac{2\sqrt{2}\;a\,C\,r\left(a^{\text{`}}-2\,M\,r+r^{\text{`}}\right)\texttt{DiracDelta}[r-r0]\,Y^{(\text{`},\text{И`},\text{И`},\text{Иt},\text{И`})}\left[-1,l,0,\tfrac{\pi}{\text{`}},0\right]}{(-rm+rp)^{\text{`}}}$$

Finally, we can simplify the results by using identities for SpinWeightedSpherical Harmoncis

In[63]:=
```
HarmonicRules =
  {Y^(0,0,0,1,0)[s_, l_, m_, θ_, ϕ_] → s Cot[θ] Y[s, l, m, θ, ϕ] - Sqrt[l + l^2 - s - s^2] Y[1 + s, l, m, θ, ϕ] -
     I Csc[θ] Y^(0,0,0,0,1)[s, l, m, θ, ϕ], Y^(0,0,0,0,1)[s_, l_, m_, θ_, ϕ_] → I m Y[s, l, m, θ, ϕ],
   Conjugate[Y[s_, l_, m_, θ_, ϕ_]] → (-1)^(-m+s) Y[-s, l, -m, θ, ϕ]};
```

Applying these rules we get the final solution

In[64]:=
```
solution /. HarmonicRules /. Y^(0,0,0,0,1)[-1, l, 0, π/2, 0] → 0;
% /. {Y[1, l, 0, π/2, 0] → -Conjugate[Y[-1, l, 0, π/2, 0]]} // Simplify;
Print["Solution == ", %]
```

$$\texttt{Solution == }-\frac{2\,i\,\sqrt{2}\;a^{2}\,C\left(a^{2}-2\,M\,r+r^{2}\right)\texttt{Conjugate}\left[Y\left[-1,l,0,\tfrac{\pi}{2},0\right]\right]\texttt{DiracDelta}[r-r0]}{(-rm+rp)^{2}}+$$

$$\frac{2\sqrt{2}\;a\,C\,\sqrt{l+l^{2}}\;r\left(a^{2}-2\,M\,r+r^{2}\right)\texttt{DiracDelta}[r-r0]\,Y\left[0,l,0,\tfrac{\pi}{2},0\right]}{(-rm+rp)^{2}}-$$

$$\frac{1}{2\sqrt{2}\,(-rm+rp)^{2}}\left(a^{2}-2\,M\,r+r^{2}\right)\texttt{Conjugate}\left[Y\left[-1,l,0,\frac{\pi}{2},0\right]\right]$$

$$\Big(-8\,i\,a^{2}\,C\,\texttt{DiracDelta}[r-r0]+\sqrt{2}\,\Big(2\,i\,\sqrt{2}\;C\,r^{2}\,\texttt{DiracDelta}[r-r0]+2\,i\,\sqrt{2}$$

$$C\left(a^{2}+r^{2}\right)\texttt{DiracDelta}[r-r0]+i\,\sqrt{2}\;C\,r\left(a^{2}+r^{2}\right)\texttt{DiracDelta}'[r-r0]\Big)\Big)$$

Since the solution given in the article (C = I $\left(\frac{\Delta 0}{P0}\right)$) and (e =
0)

$$
\begin{aligned}
{}^2J_{lm} = &- \frac{\Delta\delta_{m0}}{\sqrt{2}\,(r_+ - r_-)^2}\ [(Mae/\mathfrak{A}_0) + \pi\ \mathfrak{J}(\Delta_0/\mathfrak{A}_0)^{1/2}] \\
&\cdot \Bigg[i(r_0^2 + a^2)\ {}_{-1}\overline{Y}_{l0}\left(\frac{\pi}{2}, 0\right)\ \delta'(r - r_0) \\
&+ \left\{ ir_0\ {}_{-1}\overline{Y}_{l0}\left(\frac{\pi}{2}, 0\right) - a\,[l(l+1)]^{1/2}\ {}_0\overline{Y}_{l0}\left(\frac{\pi}{2}, 0\right) \right\} \delta\,(r - r_0) \Bigg]
\end{aligned}
$$

Is only quadratic in r, but our solution is at least quartic in r.