First we define known quantities

In[∘]:=
```
Δ := r^2 - 2 M r + a^2
Σ := r^2 + a^2 Cos[θ]^2
J := C DiracDelta[r - r0] DiracDelta[θ - π/2]
Jm† := (√2 (r - ⅈ a Cos[θ]))^-1 + ⅈ (r^2 + a^2) Sin[θ] J
Jn := - (a Δ)/Σ Sin[θ]^2 J
```

Now for J2, we can ignore ∂_φ since Jm and Jn are independent of it

$$J_2 = \frac{-\Delta}{2\sqrt{2}\,\Sigma\,(r - ia\cos\theta)^2}\left[\sqrt{2}\left(\frac{\partial}{\partial r} - \frac{a}{\Delta}\frac{\partial}{\partial\varphi} + \frac{1}{r - ia\cos\theta}\right)(r - ia\cos\theta)^2 J_{\overline{m}} + 2\left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi} + \frac{ia\sin\theta}{r - ia\cos\theta}\right)\frac{\Sigma\,(r - ia\cos\theta)}{\Delta}J_n\right] \tag{2.12}$$

In[∘]:=
```
J2 := -Δ/(2 √2 Σ (r - ⅈ a Cos[θ])^2)
  (√2 (∂_r # + 1/(r - ⅈ a Cos[θ]) #) &@((r - ⅈ a Cos[θ])^2 Jm†) + 2 (∂_θ # + (ⅈ a Sin[θ])/(r - ⅈ a Cos[θ]) #) &@
   ((Σ (r - ⅈ a Cos[θ]))/Δ Jn))
```

Finally for the integration we need to first look at the definition of the SpinWeighted spherical harmonics.

Since all SpinWeighted spherical harmonics, here denoted ${}_sY_{\mathrm{lm}}(\theta,\phi)$= Y[s,l,$\theta$,$\phi$] behave in the ϕ argument as Exp[ⅈ m ϕ] and since all other variables are independent of ϕ in the equation above We have

$\int$Y[s,l,m,θ,ϕ]= 2 π DiracDelta[m,0] Y[s,l,m,θ,0]. Then we can integrate over θ.

e the source term is given by

$$^2J_{lm}(r) = \int_0^{2\pi}\int_0^{\pi}\frac{(r - ia\cos\theta)^2}{(r_+ - r_-)^2}\;\Sigma\,J_2\;{}_{-1}\overline{Y}_{lm}\,\sin\theta\,d\theta\,d\varphi$$

In[∘]:=

We can split the integrand into two parts depending if we have DiracDelta$\left[\theta - \frac{\pi}{2}\right]$ or the derivative DiracDelta'$\left[\theta - \frac{\pi}{2}\right]$.

and we will use the standard properties

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,d\,x = f(a)$$

$$\int f(x)\,\delta^{(n)}(x)\,d\,x \equiv -\int \frac{\partial f}{\partial x}\,\delta^{(n-1)}(x)\,d\,x.$$

First part

In[∘]:= $\frac{(r - i\, a\, \mathrm{Cos}[\theta])^2}{(rp - rm)^2}$ Σ J2 Conjugate[Y[-1, l, 0, θ, 0]];

% /. DiracDelta[-π + 2 θ] → 1 /. DiracDelta′[-π + 2 θ] → 0;

FirstPart = % /. θ → $\frac{\pi}{2}$

Out[∘]= $-\frac{1}{2\sqrt{2}\,(-rm + rp)^2}\left(a^2 - 2\,M\,r + r^2\right)\mathrm{Conjugate}\left[Y\left[-1,\, l,\, 0,\, \frac{\pi}{2},\, 0\right]\right]$
$\left(-8\,i\,a^2\,C\,\mathrm{DiracDelta}[r - r0] + \sqrt{2}\left(3\,r\left(\frac{1}{\sqrt{2}\,r} + 2\,i\,C\left(a^2 + r^2\right)\mathrm{DiracDelta}[r - r0]\right) + \right.\right.$
$\left.\left. r^2\left(-\frac{1}{\sqrt{2}\,r^2} + 4\,i\,C\,r\,\mathrm{DiracDelta}[r - r0] + 2\,i\,C\left(a^2 + r^2\right)\mathrm{DiracDelta}'[r - r0]\right)\right)\right)$

Second part

In[∘]:= $\frac{(r - i\, a\, \mathrm{Cos}[\theta])^2}{(rp - rm)^2}$ Σ J2 Conjugate[Y[-1, l, 0, θ, 0]];

% /. DiracDelta[-π + 2 θ] → 0 /. DiracDelta′[-π + 2 θ] → 1;

SecondPart = -D[%, θ] /. θ → $\frac{\pi}{2}$ /. Conjugate′[Y[-1, l, 0, $\frac{\pi}{2}$, 0]] → 1

Out[∘]= $-\frac{2\,i\,\sqrt{2}\,a^2\,C\left(a^2 - 2\,M\,r + r^2\right)\mathrm{Conjugate}\left[Y\left[-1,\, l,\, 0,\, \frac{\pi}{2},\, 0\right]\right]\mathrm{DiracDelta}[r - r0]}{(-rm + rp)^2} +$
$\frac{\left(a^2 - 2\,M\,r + r^2\right)\left(2 - 8\,a\,C\,r\,\mathrm{DiracDelta}[r - r0]\right)Y^{(0,0,0,1,0)}\left[-1,\, l,\, 0,\, \frac{\pi}{2},\, 0\right]}{2\sqrt{2}\,(-rm + rp)^2}$

We get the solution

In[∘]:=
```
FirstPart + SecondPart // Simplify
```

Out[∘]=

$$-\frac{2\,i\,\sqrt{2}\,a^2\,C\left(a^2-2\,M\,r+r^2\right)\text{Conjugate}\left[Y\left[-1,\,l,\,0,\,\frac{\pi}{2},\,0\right]\right]\text{DiracDelta}[r-r0]}{(-rm+rp)^2}-\frac{1}{2\,\sqrt{2}\,(-rm+rp)^2}\left(a^2-2\,M\,r+r^2\right)\text{Conjugate}\left[Y\left[-1,\,l,\,0,\,\frac{\pi}{2},\,0\right]\right]\left(-8\,i\,a^2\,C\,\text{DiracDelta}[r-r0]+\sqrt{2}\left(3\,r\left(\frac{1}{\sqrt{2}\,r}+2\,i\,C\left(a^2+r^2\right)\text{DiracDelta}[r-r0]\right)+r^2\left(-\frac{1}{\sqrt{2}\,r^2}+4\,i\,C\,r\,\text{DiracDelta}[r-r0]+2\,i\,C\left(a^2+r^2\right)\text{DiracDelta}'[r-r0]\right)\right)\right)+\frac{\left(a^2-2\,M\,r+r^2\right)\left(2-8\,a\,C\,r\,\text{DiracDelta}[r-r0]\right)Y^{(0,0,0,1,0)}\left[-1,\,l,\,0,\,\frac{\pi}{2},\,0\right]}{2\,\sqrt{2}\,(-rm+rp)^2}$$

We should further utilize identities for the Y function but we can already see that we must have made a mistake

Since the solution given in the article ($C = I\left(\frac{\Delta 0}{P0}\right)$) and ($e = 0$)

$$^2J_{lm} = -\frac{\Delta\delta_{m0}}{\sqrt{2}\,(r_+ - r_-)^2}\;\left[(Mae/\mathcal{P}_0) + \pi\,\mathcal{I}(\Delta_0/\mathcal{P}_0)^{1/2}\right] \cdot\left[i(r_0^2+a^2)\;{}_{-1}\overline{Y}_{l0}\left(\frac{\pi}{2},0\right)\,\delta'(r-r_0) + \left\{ir_0\;{}_{-1}\overline{Y}_{l0}\left(\frac{\pi}{2},0\right) - a\,[l(l+1)]^{1/2}\;{}_0\overline{Y}_{l0}\left(\frac{\pi}{2},0\right)\right\}\delta(r-r_0)\right]$$

Is only quadratic in r, but our solution is at least quartic in r.