

We start with the heat equation:

$$\frac{\delta T}{\delta t} - \kappa \Delta^2 T = 0$$

and divide it into :

$$u(x, y, z, t) = u(x, y, z) T(t)$$

$$u_t = \kappa \Delta^2 u$$

The problem for each of  $X(x)$ ,  $Y(y)$ ,  $Z(z)$  is the 1D Sturm-Liouville problem:

$$X^{(2)}(x) + \mu_1 X(x) = 0; 0 \leq x \leq a; X(0) = X(a) = 0$$

$$X_l(x) = a_l \sin\left(\frac{l\pi x}{a}\right)$$

with eigenvalue:  $\lambda_1 = \left(\frac{l\pi}{a}\right)^2$

Similarly we can solve the  $Y(y)$  and  $Z(z)$  equations:

$$Y^{(2)}(y) + \mu_2 Y(y) = 0; 0 \leq y \leq a; Y(0) = Y(a) = 0$$

$$Y_m(y) = a_m \sin\left(\frac{m\pi y}{a}\right) \quad \text{with eigenvalue: } \lambda_2 = \left(\frac{m\pi}{a}\right)^2$$

$$Z^{(2)}(z) + \mu_3 Z(z) = 0; 0 \leq z \leq a; Z(0) = Z(a) = 0$$

$$Z_n(z) = a_n \sin\left(\frac{n\pi z}{a}\right) \quad \text{with eigenvalue: } \lambda_3 = \left(\frac{n\pi}{a}\right)^2$$

The eigen solution of the 3D Sturm Liouville problem is:

$$v_{lmn} = A_{lmn} \sin\left(\frac{\pi l}{a} x\right) \sin\left(\frac{\pi m}{a} y\right) \sin\left(\frac{\pi n}{a} z\right)$$

with eigen value:

$$\lambda_{lmn} = \left(\frac{l^2}{a^2} + \frac{m^2}{a^2} + \frac{n^2}{a^2}\right) \pi^2$$

The solution for  $T(t)$  is thus given by:

$$T_{lmn}(t) = e^{-\lambda_{lmn} \kappa t} = e^{-\left(\frac{l^2}{a^2} + \frac{m^2}{a^2} + \frac{n^2}{a^2}\right) \pi^2 \kappa t}$$

Following from the 1 D solution it is shown that:

$$v_{lmn}(x, y, z, t) = v_{lmn}(x, y, z) T_{lmn}(t)$$

$$v(x, y, z, t) = \sum_{l, m, n} A_{lmn} \sin\left(\frac{\pi l}{a} x\right) \sin\left(\frac{\pi m}{a} y\right) \sin\left(\frac{\pi n}{a} z\right) e^{-\left(\frac{l^2 + m^2 + n^2}{a^2}\right) \pi^2 \kappa t}$$

To solve for the initial conditions we take the initial temperature at  $T_0$  and the heat reservoir at 0. we apply take the sum over integrals from 0 to 1, and use a kronecker delta function to solve for each of  $l, m, n$ :

$$\begin{aligned}
A_{lmn} &= \left( 2 \int_0^a \sin\left(\frac{l\pi x}{a}\right) f(x) dx \right) \left( 2 \int_0^a \sin\left(\frac{m\pi y}{a}\right) f(y) dy \right) \left( 2 \int_0^a \sin\left(\frac{n\pi z}{a}\right) f(z) dz \right) \\
A_{lmn} &= T_0 \left( -2 \left[ \frac{\cos(l\pi) - 1}{l\pi} \right] \right) \left( -2 \left[ \frac{\cos(m\pi) - 1}{m\pi} \right] \right) \left( -2 \left[ \frac{\cos(n\pi) - 1}{n\pi} \right] \right) \\
A_{lmn} &= T_0 \left( \frac{4}{l\pi} \right) \left( \frac{4}{m\pi} \right) \left( \frac{4}{n\pi} \right) \\
A_{lmn} &= T_0 \frac{64}{lmn \pi^3}
\end{aligned}$$

The final solution therefore has the form:

$$T = T_0 \sum_{l,m,n} \frac{64}{lmn \pi^3} \sin\left(\frac{\pi l}{a} x\right) \sin\left(\frac{\pi m}{a} y\right) \sin\left(\frac{\pi n}{a} z\right) e^{-\left(\frac{l^2 + m^2 + n^2}{a^2}\right) \pi^2 \kappa t}$$

For initial temperature at  $T_1$ :

$$T = (T_0 - T_1) \sum_{l,m,n} \frac{64}{lmn \pi^3} \sin\left(\frac{\pi l}{a} x\right) \sin\left(\frac{\pi m}{a} y\right) \sin\left(\frac{\pi n}{a} z\right) e^{-\left(\frac{l^2 + m^2 + n^2}{a^2}\right) \pi^2 \kappa t}$$