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Rotating Coordinates as Tools for Calculating Circular Geodesics and Gyroscopic Precession

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If an axially symmetric stationary metric is given in standard form (i.e. in coordinates adapted to the symmetries) the transformation $\phi \mapsto \phi' = \phi - \omega t$ (ω = constant) of the azimuthal angle leads to another such standard form. The spatial lattice L' corresponding to the latter rotates at angular velocity ω relative to the lattice L of the former. For the standard form of a stationary metric there are simple formulae giving the four-acceleration of a given lattice point and the rotation of a gyroscope at a given lattice point. Applying these formulae to L', we find the condition for circular paths about the axis in L to be 4-geodesic, and also the precession of gyroscopes along circular paths which are not necessarily geodesic. Among other examples we re-obtain the complete geodesic structure of the Gödel universe, and the gyroscopic precessions associated with the names of Thomas, Fokker and de Sitter, and Schiff.

1. INTRODUCTION: THE METHODOLOGY

As is well known (see Ref. 1, p. 250, Ref. 2, p. 183), the metric of any stationary space-time can be cast into the canonical form

$$\mathrm{ds}^2 = -e^{2\psi}(dt - w_i dx^i)^2 + h_{ij}dx^i dx^j, \tag{1}$$

with $i = 1, 2, 3$, where ψ, w_i, and h_{ij} depend on the spatial coordinates x^i only. (We use units making $c = G = 1$ throughout.) In fact, the

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admitting of such a coordinate representation can serve as the *definition* of a stationary space-time. The representation (1) is form-invariant under (i) the group of spatial transformations $x'^i = x'^i(x^1, x^2, x^3)$ and (ii) the group of time transformations

$$t' = a[t + f(x^1, x^2, x^3)], \tag{2}$$

where a is an arbitrary constant and f an arbitrary differentiable function. Under (i) the quantities h_{ij}, w_i, $\psi_{,i}$, $w_{[i,j]}$ behave as 3-tensors (we write "$_{,i}$" for $\partial/\partial x^i$); under (ii) ψ and w_i transform as follows:

$$\psi \mapsto \psi' = \psi - \ln a, \quad w_i \mapsto w'_i = a(w_i + f_{,i}), \tag{3}$$

which leaves $\psi_{,i}$ and $e^{\psi} w_{[i,j]}$ invariant, and of course h_{ij} also. By a suitable time transformation (2) we can therefore achieve $\psi = 0$ and $w_i = 0$ at any preassigned event without affecting h_{ij}. Thus

$$dl^2 = h_{ij} dx^i dx^j \tag{4}$$

is recognized as the metric of the "lattice" of "fixed points" or "reference particles" $x^i =$ constant.

Two straightforward calculations (see Appendix) lead to the following two pairs of formulae for the four-acceleration A_μ and the rotation three-vector $\boldsymbol{\Omega}$ of the congruence of world-lines $x^i =$ constant (with $\mu, \nu = 0, 1, 2, 3$ and $x^o = t$):

$$A_\mu = (0, \operatorname{grad} \psi) = (0, \psi_{,i}), \tag{5}$$

$$A = (g^{\mu\nu} A_\mu A_\nu)^{1/2} = (h^{ij} \psi_{,i} \psi_{,j})^{1/2}, \tag{6}$$

and

$$\boldsymbol{\Omega} = \tfrac{1}{2} e^{\psi} \operatorname{curl} \mathbf{w}, \quad \text{i.e., } \Omega^i = \tfrac{1}{2} e^{\psi} (\det h_{mn})^{-1/2} \epsilon^{ijk} w_{k,j}, \tag{7}$$

$$\Omega = (h_{ij} \Omega^i \Omega^j)^{1/2} = e^{\psi} (h^{ij} h^{kl} w_{[i,k]} w_{[j,l]})^{1/2}. \tag{8}$$

We note that $\operatorname{grad} \psi$ is the proper three-acceleration of a reference particle $x^i =$ constant, i.e. its acceleration in the local inertial rest frame. In particular, and importantly, if $\Psi_{,i} = 0$, then $A_\mu = 0$ and the reference particle describes a geodesic. The vector $\boldsymbol{\Omega}$, on the other hand, describes the rate of rotation with respect to proper time at any reference particle P, of the set of neighboring reference particles, relative to the local compass of inertia. *Mathematically* that compass corresponds to a triad of spatial vectors Fermi-Walker transported along P's world-line. *Physically* it can be realized by a triad of gyroscopes spinning about three mutually orthogonal axes. Of course, the proper angular velocity of the compass of inertia relative to the reference particles is given by $-\mathbf{\Omega}$.

In the present paper we utilize these more or less familiar results for two purposes, namely (i) to find circular geodesics, and (ii) to find the precession of gyroscopes along circular paths, in stationary space-times with axial symmetry (i.e. with "circular" Killing vector fields). Examples of such space-times are those associated with the names of Minkowski, Schwarzschild, Reissner-Nordstrøm, Weyl, Lewis, Gödel, Kerr, Kerr-Newman, Tomimatsu-Sato, etc. The standard method is to solve the geodesic equation in the first case, and to Fermi-Walker transport a triad around the orbit in the second case. Our method consists in replacing the azimuthal coordinate ϕ of the given metric (when written in polar coordinates about an axis of symmetry) by another, ϕ', such that

$$\phi = \phi' + \omega t \qquad (\omega = \text{ constant}) \tag{9}$$

(see Fig. 1), and leaving all other coordinates, in particular also the time t, unchanged. This transforms the original coordinate representation of the stationary metric into another, based on the same coordinate time, whose lattice of fixed points r, θ, $\phi' =$ constant (or r, z, $\phi' =$ constant in the case of cylindrical polar coordinates) rotates relative to the first at angular velocity ω with respect to coordinate time t. It is to the *rotating* lattice that we apply the formulae (5) and (8). In particular, setting the new $\psi_{,i}$ [or, equivalently, $(e^{2\psi})_{,i}$] equal to zero gives us the condition that a fixed new lattice point describes a geodesic, which translates, of course, into a circular geodesic in the original lattice. (This process happens to yield *all* the timelike and null geodesics in Gödel space, see Section 2 below.)

A similar approach using formula (6) yields the (nongravitational) centrifugal or centripetal force, $\operatorname{grad}\psi$, that would have to be applied in order to keep a particle in a nongeodesic circular orbit.

Lastly, evaluating $-\mathbf{\Omega}$ in the rotating system gives us the rate of rotation of a compass of inertia at a point fixed in the rotating lattice; when this is suitably reduced by ω it yields the precession of a gyroscope that is moving uniformly round a circular orbit (geodesic or not) in the original lattice. In this way one easily calculates, for example, the well-known Thomas precession in Minkowski space, the Fokker-de Sitter precession in Schwarzschild space, and the additional Schiff precession in Kerr space (see Section 3 below).

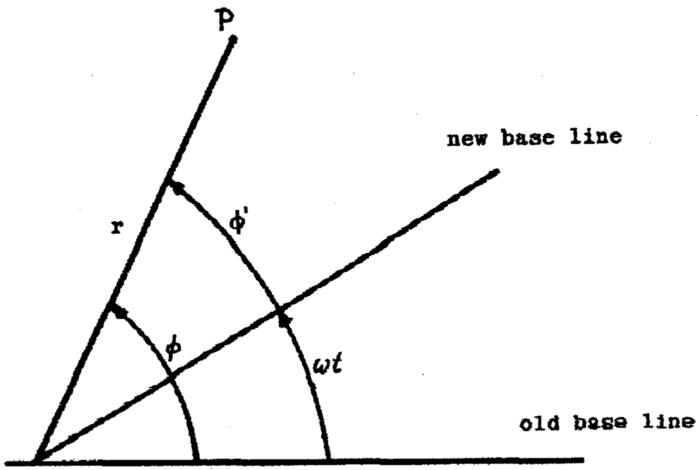

Fig. 1.
Old and new polar coordinates of a point P: (r, ϕ), (r, ϕ').

2. CIRCULAR GEODESICS

In this section we exemplify our method by finding the circular geodesics in the Schwarzschild,[3] Kerr, and Gödel space-times.

2.1 Schwarzschild Space

The well-known standard form of the Schwarzschild metric (describing the vacuum space-time around a spherically symmetric mass distribution) is as follows:

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (10)$$

If we apply to this the transformation (9) and then cast the resulting representation of the metric into the canonical form (1), we obtain, after setting $\theta = \pi/2$,

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r} - r^2\omega^2\right)\left(dt - \frac{r^2\omega}{1-(2m/r)-r^2\omega^2}\,d\phi'\right)^2 + \left(1-\frac{2m}{r}\right)^{-1} dr^2 + \frac{r^2-2mr}{1-(2m/r)-r^2\omega^2}\,d\phi'^2. \quad (11)$$

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[3] Our method for obtaining the circular geodesics in Schwarzschild space has already been used at the suggestion of one of us (W. R.) by A. Rosenblum [3].

Setting $\theta = \pi/2$ is justified by the fact that the surface $\theta = \pi/2$ is a "symmetry surface" and thus contains any geodesic that touches it; by the spherical symmetry of Schwarzschild space all geodesics therefore lie in the surface $\theta = \pi/2$ or in one of the isometric surfaces into which $\theta = \pi/2$ can be "rotated" about the origin.

The first parenthesis in (11) corresponds to the $e^{2\psi}$ of the canonical form (1), here a function of r only. Setting its r-derivative equal to zero at once yields

$$\omega^2 = mr^{-3}, \tag{12}$$

the well-known "exact Kepler law" governing circular geodesics in Schwarzschild space, with $\omega = d\phi/dt$. A simple check shows that these are timelike as long as $r > 3m$. In the limit, at $r = 3m$, we have a circular light path.

2.2 Kerr Space

Next we use the same method to find the circular geodesics in the equatorial plane of the Kerr metric. In Boyer-Lindquist coordinates [4], and restricted to the equatorial plane $\theta = \pi/2$ (which again is a symmetry surface), this metric reads

$$\begin{aligned} \mathrm{ds}^2 = &-\left(1 - \frac{2m}{r}\right) dt^2 - \frac{4am}{r}\, dt\, d\phi + \left(r^2 + a^2 + \frac{2a^2 m}{r}\right) d\phi^2 \\ &+ \left(1 + a^2 r^{-2} - \frac{2m}{r}\right)^{-1} dr^2. \end{aligned} \tag{13}$$

It describes the space-time geometry outside a material object with mass m and angular momentum am (a and m both having the dimension of a length here), which has undergone gravitational collapse. But it also applies, at least approximately, to finite, spherical, rotating masses. We require $0 \le a \le m$ (the limiting case $a = 0$ leads us back to the Schwarzschild metric) so as to avoid the appearance of a naked singularity.

In order to find the circular geodesics of this metric outside the horizon $r = r_+ = m + (m^2 - a^2)^{1/2}$, we apply the azimuthal transformation (9) to (13), which results in

$$\begin{aligned} \mathrm{ds}^2 = &-\left[1 - \omega^2(a^2 + r^2) - \frac{2m}{r}(1 - a\omega)^2\right] \times \\ &\times \left[dt - \frac{\omega(r^2 + a^2) - (2am/r)(1 - a\omega)}{1 - \omega^2(r^2 + a^2) - (2m/r)(1 - a\omega)^2}\, d\phi'\right]^2 \\ &+ \frac{r^2 - 2mr + a^2}{1 - \omega^2(r^2 + a^2) - (2m/r)(1 - a\omega)^2}\, d\phi'^2 \\ &+ \left(1 + a^2 r^{-2} - \frac{2m}{r}\right)^{-1} dr^2. \end{aligned} \tag{14}$$

From this we read off the new $e^{2\psi}$ to be

$$e^{2\psi} = 1 - \omega^2(a^2 + r^2) - \frac{2m}{r}(1 - a\omega)^2. \tag{15}$$

Setting its r-derivative equal to zero gives us the geodesic condition

$$\omega = \left(a \pm \sqrt{\frac{r^3}{m}}\right)^{-1}. \tag{16}$$

(The a in this formula represents an "inertial-frame-dragging" correction to the Kepler law.) For each value of $r > r_+$ there is one positive and one negative value of ω, since then $a < \sqrt{r^3/m}$. However, the corresponding geodesic is timelike only if

$$\mp 2am\sqrt{\frac{r^3}{m}} < r^2(r - 3m). \tag{17}$$

Hence, for large r, both the forward geodesic (corresponding to the upper sign) and the retrograde geodesic (corresponding to the lower sign) are timelike. When r reaches the largest root r_1 of the equation

$$r(r - 3m)^2 - 4a^2m = 0 \tag{18}$$

(a number greater than or equal to $3m$), the retrograde geodesic becomes lightlike, whereas the forward geodesic remains timelike until r reaches the second root r_2 of (18) (a number between m and $3m$). If we go to the Schwarzschild limit by setting a equal to zero, r_1 and r_2 amalgamate at $3m$. In the "extreme Kerr" case $a = m$, the corresponding radii are given by $r_1 = 4m$ and $r_2 = m$. It is interesting to note that the latter is already situated inside the ergosphere, i.e. the region between $r = r_+$ and $r = 2m$, where t in the original representation (13) of the metric is no longer a time coordinate. In fact, $r = 2m$ is the boundary of stationarity relative to infinity. Nevertheless, and this justifies our method, the *rotating* lattice is stationary down to $r = r_+$ for sufficiently large ω; it has an *outer* limit of stationarity, where the orbital speed becomes c.

2.3 Gödel Space

As our third example, we consider the Gödel universe. One form of its metric suitable for our purpose is[4]

$$ds^2 = -4R^2[(dt + \sqrt{2}\,S^2 d\phi)^2 - \{dr^2 + (S^2 + S^4)d\phi^2 + dz^2\}], \tag{19}$$

[4] See Ref. 4, p.169. (Our lattice rotates in the opposite sense from theirs.)

where $S = \sinh r$ and R is a constant length. This stationary metric satisfies Einstein's field equations with dust of constant density $\rho = 1/(8\pi R^2)$ and a cosmological constant $\Lambda = -1/(2R^2)$. Its lattice can be visualized as an infinite stack of surfaces $z =$ constant, all having negative constant curvature $-R^{-2}$, as can be seen by casting their metric into the form $R^2(dr'^2 + \sinh^2 r' d\phi^2)$, with $r' = 2r$. (Note that $S^2 + S^4 = S^2C^2 = \frac{1}{4}\sinh^2 2r$, with $C = \cosh r$.) The lattice has translational symmetry in the z-direction as well as rotational symmetry about any z-line. Gödel's universe is thus homogeneous both in space and in time. All lattice points describe geodesics [as can be seen from the constancy of "$e^{2\psi}$"; cf. eqs. (1) and (5)], and the lattice rotates everywhere at proper angular velocity $\Omega = -(\sqrt{2}\,R)^{-1}$ about the z-direction relative to the compass of inertia [cf. eqs. (1), (7) and (8)]. (We shall take Ω to be negative if the sense of Ω is opposite to that of increasing ϕ.) Thus we seem to have a uniformly rotating rigid cylinder which is infinite in every direction, though there is, of course, no extended inertial frame "in" which this rotation takes place.

Applying to (19) the azimuthal transformation (9) and then "completing the square" to arrive once more at a canonical form (1), we find

$$ds^2 = -4R^2[1 + 2\sqrt{2}\,S^2\omega + (S^4 - S^2)\omega^2]\,(dt + ...)^2 + ... \,, \qquad (20)$$

where the ellipses stand for terms of no relevance to our present purpose. The expression preceding the dt-term corresponds to the $e^{2\psi}$ of the new canonical form and depends on r only. Accordingly we set its r-derivative equal to zero in order to obtain the equation of circular geodesics in the symmetry surfaces $z =$ constant:

$$4\sqrt{2}\,SC\omega + (4S^3C - 2SC)\omega^2 = 0. \qquad (21)$$

The root $\omega = 0$ should not surprise us: as we have seen, the original lattice points themselves describe geodesics. After elimination of this "trivial" solution, equation (21) yields

$$\omega = \frac{2\sqrt{2}}{(1 - 2S^2)} \qquad (22)$$

as the equation governing circular geodesics. Note that, as r increases from zero, ω increases from $2\sqrt{2}$: larger orbits have lesser periods. A little surprisingly, perhaps, the proper angular velocity of infinitesimally small orbits [to convert to *proper* angular velocity, we must multiply by $(2R)^{-1}$] corresponds to *twice* that of the compass of inertia. The given mass distribution seems to allow identical small orbits in either sense *relative to the local inertial rest frame.*

To be of physical interest, an orbit must be timelike or null. Since the ellipses in (20) stand for terms involving the spatial differentials, we must now set them equal to zero, substitute for ω from (22), and demand $\mathbf{ds}^2 \le 0$. This leads to the condition

$$1 - 4S^2 - 4S^4 \geq 0, \tag{23}$$

or, equivalently,

$$\sinh^2 2r \leq 1. \tag{24}$$

Numerically, (24) corresponds to

$$r \leq r_o := \tfrac{1}{2}\ln(1+\sqrt{2}) = 0.44068679\ldots\,, \tag{25}$$

and also to $S^2 \leq \frac{1}{2}(\sqrt{2}-1) = 0.2071067\ldots$, which leads via (22) to

$$2\sqrt{2} < \omega \leq 2(\sqrt{2}-1)^{-1} =: \omega_o = 4.8284271\ldots \tag{26}$$

Note that, unlike the Keplerian orbits (12), all present orbits have the same sense, namely that of ϕ increasing; but corresponding to every "right-handed" Gödel universe there is a "left-handed" one ($\phi \mapsto -\phi$).

Circular *light* orbits in the surfaces $z =$ constant have coordinate radius r_o (corresponding to proper radius $2Rr_o$) and coordinate-time period $2\pi/\omega_o$ (corresponding to proper time period $4\pi R/\omega_o$). But by the homogeneity and isotropy of these surfaces, all light rays in them must be such circles, in particular all those emanating from a given event in all directions satisfying $z =$ constant. This set of rays consequently re-converges to the original lattice point at coordinate time $2\pi/\omega_o$ later. Similarly, all timelike geodesics orthogonal to a z-line are circles, their radii r being restricted by $0 \leq r < r_o$, and their coordinate time periods being given by $2\pi/\omega = \pi(1-2S^2)/\sqrt{2}$.

It is clear from the original representation (19) of the metric that each geodesic motion in the Gödel universe is the composition of a geodesic motion in a surface $z =$ constant with a *uniform* motion in the z-direction (z a linear function of t). For a direct proof of this fact we can augment the azimuthal transformation (9) by

$$z = z' + vt \qquad (v = \text{ constant}), \tag{27}$$

thus letting the "new" lattice not only rotate at angular velocity ω, but also translate at velocity v. This brings an additional term $-v^2$ into the square bracket in (20), which, however, has no effect on the result (21).

Consequently a point fixed in the new lattice (now *spiraling* in the z-direction) describes a geodesic when (22) is satisfied, precisely as before. (A similar argument is, of course, possible whenever there is translational symmetry in the direction of an axis of rotational symmetry.) Free particles here move somewhat like charged particles in a parallel magnetic field. As v, the velocity component in the z-direction, increases, the maximum radius of the spiral, for timelikeness, decreases until it shrinks to zero for $v = c$, for the condition of timelikeness is now

$$\frac{1-4S^2}{4S^4} > \frac{1+v^2}{1-v^2}. \tag{28}$$

Evidently we have re-obtained the full geodesic structure of the Gödel universe by the new method: every initial motion leads to one of the above circles or spirals.

3. GYROSCOPIC PRECESSION

We now present four examples of the use of rotating coordinates in determining gyroscopic precession: the Thomas precession, the Fokker-de Sitter precession, the Schiff precession, and the precession in Gödel space.

3.1 The Thomas precession in Minkowski space

The Thomas precession [5] refers to the precession of the compass of inertia along an arbitrary path in Minkowski space. For brevity and vividness one often loosely speaks of the precession of "a gyroscope" instead. Here we shall examine the most frequently contemplated path, a full circle. (Thomas himself discussed the orbital motion of a spinning electron in an atom.)

By setting $m = 0$ in the metric (11) for Schwarzschild space we obtain the metric representation corresponding to a rotating coordinate lattice in Minkowski space, once again restricted to the symmetry surface $\theta = \pi/2$:

$$\mathrm{d}s^2 = -(1-r^2\omega^2)\left(dt - \frac{r^2\omega}{1-r^2\omega^2}\,d\phi'\right)^2 + dr^2 + \frac{r^2}{1-r^2\omega^2}\,d\phi'^2. \tag{29}$$

Preparatory to applying formulae (7) and (8), we note that for the metric (29), with the choice $x^1, x^2, x^3 = r, \theta, \phi'$ (so as to have right-handed coordinates), and the notation $v = r\omega$, we have, by comparison with the metric (1),

$$\begin{aligned} &e^{2\psi} = 1 - v^2, \qquad w_3 = rv(1-v^2)^{-1}, \qquad w_{3,1} = 2v(1-v^2)^{-2} \\ &h^{11} = 1, \qquad h^{33} = r^{-2}(1-v^2). \end{aligned} \tag{30}$$

Accordingly, (7) shows that $\boldsymbol{\Omega}$ points in the positive direction orthogonal to the plane of the orbit, and (8) yields for its magnitude

$$\Omega = e^{\psi}(h^{11}h^{33}w^2_{[3,1]})^{1/2} = \frac{v}{r}(1-v^2)^{-1}, \tag{31}$$

without approximation. Now we recall that $-\boldsymbol{\Omega}$ gives the precession of the gyroscope relative to the lattice. So after one complete revolution the orientation of the gyroscope changes by an angle

$$\begin{aligned}\Delta\phi' &= -\Omega\Delta\tau = -\Omega e^{\psi}\Delta t = -\Omega e^{\psi}2\pi\omega^{-1} \\ &= -\frac{v}{r}(1-v^2)^{-1}(1-v^2)^{1/2}\frac{2\pi}{\omega} = -2\pi(1-v^2)^{-1/2}.\end{aligned} \tag{32}$$

Without loss of generality, we here assume $\omega > 0$. Of course, the rotating coordinate system precesses the baseline by 2π per revolution, and this must be allowed for in obtaining the precession per revolution relative to the original system:

$$\Delta\phi = \Delta\phi' + 2\pi = -2\pi[(1-v^2)^{-1/2} - 1] \approx -\pi v^2. \tag{33}$$

This is Thomas' well-known result, the negative sign indicating that the precession is retrograde.

3.2 The Fokker-de Sitter precession in Schwarzschild space

The Fokker-de Sitter precession [6,7] refers to the precession of a gyroscope following a free (i.e. geodesic) orbit, generally taken to be a circle, around a spherically symmetric mass, such as the earth. It is of renewed interest in connection with various gyroscopic tests for general relativity currently in the planning stage. But, in fact, for the earth-moon "gyroscope" in orbit around the sun, it has already been verified to an astonishing accuracy of 2% [11]. The relevant metric is the Schwarzschild metric (10), which for our purposes we again refer to rotating coordinates, thus obtaining the form (11). We have seen already that circular Schwarzschild orbits obey Kepler's third law, eq. (12). Comparing (11) with the canonical form (1), and substituting, where convenient, from (12), we easily find

$$\begin{aligned}&e^{2\psi} = 1 - \frac{3m}{r}, \\ &w_3 = r^2\omega\left(1 - \frac{2m}{r} - r^2\omega^2\right)^{-1}, \qquad w_{3,1} = 2r\omega\left(1 - \frac{3m}{r}\right)^{-1}, \\ &h^{11} = 1 - \frac{2m}{r}, \qquad h^{33} = \left(1 - \frac{3m}{r}\right)\left(1 - \frac{2m}{r}\right)^{-1} r^{-2},\end{aligned} \tag{34}$$

where again we chose $x^1, x^2, x^3 = r, \theta, \phi'$. And again eq. (7) shows that Ω points in the positive direction orthogonal to the plane of the orbit. Substitution from (7) into (8) yields, coincidentally,

$$\Omega = e^{\psi}(h^{11}h^{33}w^2_{[3,1]})^{1/2} = \omega. \tag{35}$$

Of course, Ω is a *proper* rate of rotation while ω is a *coordinate* rate of rotation. So, repeating, mutatis mutandis, the argument that leads to (32) and (33), and again under the assumption $\omega > 0$, we find for the precession of ϕ' per revolution:

$$\Delta\phi' = -2\pi\left(1 - \frac{3m}{r}\right)^{1/2}, \tag{36}$$

without approximation, and consequently for the desired Fokker-de Sitter precession:

$$\Delta\phi = -2\pi\left[\left(1 - \frac{3m}{r}\right)^{1/2} - 1\right] \approx \frac{3\pi m}{r} = 3\pi v^2. \tag{37}$$

Because of the positive sign, the precession is in the same sense as that in which the orbit is described. [It can be shown fairly simply (Ref. 8, p. 141) that two-thirds of the precession (37) can be ascribed to the spatial geometry of the Schwarzschild metric, while one-third is essentially due to Thomas precession; however, the latter is now in the forward rather than the retrograde sense, for it is now the frame of the *field* that Thomas-precesses around the gyroscope, which itself is free, i.e. unaccelerated.]

If we repeat the above calculation *without* assuming (12), i.e. for an arbitrary (nongeodesic) circular orbit in which the required centrifugal or centripetal force grad ψ (relative to the rotating system) could be supplied, for example, by a transverse rocket, we find instead of (35) and (37):

$$\Omega = \left(1 - \frac{3m}{r}\right)\left(1 - \frac{2m}{r} - r^2\omega^2\right)^{-1}, \tag{38}$$

$$\Delta\phi = -2\pi\left[\left(1 - \frac{3m}{r}\right)\left(1 - \frac{2m}{r} - r^2\omega^2\right)^{-1/2} - 1\right]. \tag{39}$$

Note, in particular, that at radius $r = 3m$ (where the geodesic orbit would be lightlike), the precession has become so large that, independently of ω, $\Omega = 0$, i.e. the orientation of the gyroscope is locked to the lattice of the rotating frame. For smaller orbits, down to $r = 2m + \epsilon$ ($\epsilon > 0$), which can still be timelike if ω is sufficiently small, Ω becomes negative and so the precession of the gyroscope even in the rotating frame is forward, i.e. its total precession $\Delta\phi$ exceeds 2π.

Note that when we put $m = 0$ in (39), we re-obtain the Thomas precession (33).

3.3 The Schiff precession in Kerr space

Next we calculate by the same method the precession of a gyroscope that is carried along a circular geodesic in the equatorial plane of Kerr space-time. From (14) we can read off all the quantities needed for this calculation. Making use of (16), we find

$$\begin{aligned}
e^{\psi} &= \left(\pm 2a\sqrt{\frac{r^3}{m}} + \frac{r^3}{m} - 3r^2\right)^{1/2}\left(a \pm \sqrt{\frac{r^3}{m}}\right)^{-1} \\
w_{3,1} &= 2\left(a \pm \sqrt{\frac{r^3}{m}}\right)\left(r \pm \frac{ma}{r^2}\sqrt{\frac{r^3}{m}}\right)\left(\pm 2a\sqrt{\frac{r^3}{m}} + \frac{r^3}{m} - 3r^2\right)^{-1} \\
h^{11} &= 1 + \frac{a^2}{r^2} - \frac{2m}{r} \\
h^{33} &= \left(\pm 2a\sqrt{\frac{r^3}{m}} + \frac{r^3}{m} - 3r^2\right)(r^2 - 2mr + a^2)^{-1}\left(a \pm \sqrt{\frac{r^3}{m}}\right)^{-2}, \qquad (40)
\end{aligned}$$

the lower signs again corresponding to the retrograde orbits. These expressions yield for Ω, surprisingly, the same values

$$\Omega = \pm\sqrt{m/r^3} \qquad (41)$$

as in the Schwarzschild case ($a = 0$). Nonetheless, the precession $\Delta\phi'$ per revolution does depend on a, since the conversion factor from proper time to coordinate time and the angular velocity (16) involve a. We find

$$\Delta\phi' = \pm 2\pi\left(1 - \frac{3m}{r} \mp 2a\sqrt{\frac{m}{r^3}}\right)^{1/2}. \qquad (42)$$

Consequently the precession of the gyroscope with respect to the original lattice amounts to

$$\Delta\phi = \mp 2\pi\left[\left(1 - \frac{3m}{r} \pm 2a\sqrt{\frac{m}{r^3}}\right)^{1/2} - 1\right]. \qquad (43)$$

It should be emphasized that $\Delta\phi$ is the precession of the gyroscope with respect to the neighbouring points of the original lattice, and not with respect to a gyroscope situated at one of the lattice points. In the Schwarzschild case these concepts coincide, but not in the Kerr case, where the original lattice is itself rotating with respect to a compass of inertia. Within one period of coordinate time

$$\Delta t = \pm 2\pi/\omega = 2\pi(\sqrt{r^3/m} \pm a) \tag{44}$$

a gyroscope fixed at radius r in the original lattice changes its orientation with respect to neighboring points by $-2\pi am(\sqrt{r^3/m} \pm a)\,(1-2m/r)^{1/2}$, as can be easily calculated by our method. The value (43) has to be reduced by this amount, if the precession of the orbiting gyroscope is wanted with respect to a fixed gyroscope.

If both m/r and a/r are small, (43) can be replaced by the approximation

$$\Delta\phi \approx \pm\frac{3\pi m}{r} - 2\pi a\sqrt{m/r^3}. \tag{45}$$

The first term on the right gives the precession due to the mass of the central body alone, whereas the second term represents the contribution from the angular momentum of the central body. This term may properly be called the Schiff precession [9]. Note that it does not depend on the sense of the orbit. The quotientof these terms,

$$\frac{2\pi a\sqrt{m/r^3}}{3\pi m/r} = \frac{2a}{3\sqrt{mr}}, \tag{46}$$

can be viewed as a relative measure for the Schiff effect within the underlying approximation. In the case of the earth it is never more than about 1%.

3.4 Gyroscopic precession in Gödel space

As a final example, we present without detail the results of an analogous calculation for the case of Gödel's universe, restricting ourselves to the geodesic orbits for simplicity. We recall that a circular geodesic in Gödel's universe satisfies eq. (22). For a gyroscope carried along such a curve we find [following the steps that lead to (32) and to (42)] that its orientation with respect to a co-rotating lattice changes by

$$\Delta\phi' = -\pi(1-\sinh^2 2r)^{1/2} \tag{47}$$

after one revolution. This implies that its orientation with respect to the original Gödel lattice changes by

$$\Delta\phi = 2\pi - \pi(1-\sinh^2 2r)^{1/2}. \tag{48}$$

We recall [cf. eq. (24)] that the radius of timelike geodesics is restricted by the condition $0 < \sinh 2r < 1$, where the limiting cases correspond to a vanishingly small circle and to a lightlike world-line respectively.

As in the case of the Kerr metric, $\Delta\phi$ does not give the precession with respect to a second gyroscope fixed in the original Gödel lattice. In order to yield the precession in this sense, (48) has to be reduced by the precession of that second gyroscope with respect to neighboring lattice points, which amounts to $\pi(1 - 2\sinh^2 r)$. Note the necessary coincidence of this value with (48) in the limit $r \to 0$.

APPENDIX

Here we briefly justify our key formulae (5) to (8).

The 4-acceleration of a world-line $x^\mu = x^\mu(s)$ is defined as

$$A^\mu = \frac{D^2 x^\mu}{ds^2} = \frac{d^2 x}{ds^2} + \left\{ {}^{\,\mu}_{\nu\sigma} \right\} \frac{dx^\nu}{ds} \frac{dx^\sigma}{ds}, \qquad (A.1)$$

and its vanishing corresponds to the world-line's being a geodesic. Applying the alternative formula (Ref. 10, eqs. 2.431 and 2.439)

$$2A_\mu = \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} \qquad \left(\dot{x} = \frac{dx^\mu}{ds};\ L = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right) \qquad (A.2)$$

to the world-line $x^i =$ constant of the metric (1), one easily obtains (5). To obtain (6) from (5), we apply a "gauge" (time-)transformation (4) to the metric (1) to achieve $\psi = 0$, $w_i = 0$ at the event of interest, and observe that the last term in (6) is gauge invariant.

In order to establish (7), we start out from the well-known formula[5]

$$\omega_{\mu\nu} = (\delta^\sigma_\mu + V_\mu V^\sigma)(\delta^\tau_\nu + V_\nu V^\tau) V_{[\sigma,\tau]} \qquad (A.3)$$

for the vorticity tensor $\omega_{\mu\nu}$ of an arbitrary timelike congruence whose 4-velocity field is V^μ. In our case, $V^\mu = e^{-\psi}\delta^\mu_0$, whence $V_\mu = -e^\psi(\delta^0_\mu - w_i \delta^i_\mu)$, and thus

$$\omega_{kj} = e^\psi w_{[k,j]}. \qquad (A.4)$$

The components Ω^l of the rotation 3-vector $\mathbf{\Omega}$ are given by

$$\omega_{kj} = \Omega^l \eta_{ljk}, \qquad (A.5)$$

where $\eta_{ljk} = (\det h_{mn})^{1/2}\epsilon_{ljk}$ is the Levi-Civita tensor associated with the spatial metric h_{ij}. Transvecting (A.5) with $\eta^{ijk} = (\det h_{mn})^{-1/2}\epsilon^{ijk}$ yields formula (7). That (8) follows from (7) is seen most easily by transforming to Euclidean space coordinates, $h^{ij} = \delta^{ij}$, at the lattice point of interest, and noting the 3-tensorial character of the right-hand sides.

[5] See, for example, Ref. 4, eq (4.12) and after (4.1).

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