Theorem 11.2. *Let $\mathcal{G} = (\mathcal{A}, \mathcal{H}, D, C, \chi)$ be an irreducible noncommutative spin geometry, of classical dimension $n = \dim M$, over the algebra $\mathcal{A} = C^\infty(M)$ of smooth functions on a compact orientable connected manifold M without boundary. Then:*

(a) There is a unique Riemannian metric $g = g(D)$ on M, whose distance function is given by

$$d_g(x, y) = \sup\{\, |a(x) - a(y)| : a \in C(M),\ \|[D, a]\| \leq 1 \,\}.$$

(b) M is a spin manifold, and the possible operators D' for which $g(D') = g(D)$ form a finite union of affine spaces labelled by the spin structures on M.

(c) The action functional $S(D) := ⨍ |D|^{-n+2}$ yields a quadratic form on each of these affine spaces, attaining an absolute minimum at $D = \not{D}$, the Dirac operator for the corresponding spin structure; this minimum is proportional to the Einstein-Hilbert action, namely, the integral of the scalar curvature:

$$S(\not{D}) = -\frac{n-2}{24} \int_M s \sqrt{\det g}\, d^n x.$$

Before entering on the proof, the expression $⨍ |D|^{-n+2}$ requires a word of explanation. Up to now, the notation $⨍$ has been used for a certain multiple (7.83) of (any) Dixmier trace of a measurable operator. Yet the operator $|D|^{-n+2}$ does not lie in the Dixmier trace class; indeed, $|D|^{-n+2} \in \mathcal{L}^{p+}$ for $p = n/(n-2) > 1$. However, in the course of the proof, we shall show that it is actually a pseudodifferential operator. Following Connes [96], we define the **action functional** using Wodzicki residues, as follows:

$$S(D) \equiv ⨍ |D|^{-n+2} := \frac{1}{2^{\lfloor n/2 \rfloor}\Omega_n} \operatorname{Wres} |D|^{-n+2}. \tag{11.5}$$

Our normalization, chosen for compatibility with (11.2), differs from that of [96], where $⨍$ is implicitly taken to be synonymous with $n^{-1}(2\pi)^{-n}$ times the Wodzicki residue.

Our approach to the proof consists of three steps:

(1) establish the nondegeneracy of the volume form on M,

(2) build the spinor bundle over M, together with the corresponding Clifford action and Riemannian metric, and

(3) compute $S(D)$ by pseudodifferential calculus, showing that $D = \not{D} + \rho$ with $S(\not{D} + \rho)$ being a positive definite quadratic functional of a certain remainder term ρ.