For simplicity, we will consider a scalar wave in one space dimension. A simple function representing a wave moving in the positive x direction (corresponding to a plane wave in three dimensions) is given by

$$u(x,t) = e^{i(kx-\omega t)}. \tag{1}$$

This function extends over the entire x-axis. We would like to describe a wave that has finite extent, but still satisfies the wave equation. We can do this by the construction

$$f(x,t) = \int_{-\infty}^{+\infty} dk\ A(k) e^{i(kx-\omega t)}. \tag{2}$$

The amplitude $A(k)$ is the Fourier transform of $f(x,0)$, and is given by

$$A(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dx\ f(x,0) e^{-ikx}. \tag{3}$$

If $f(x,t)$ extended over all space, like $u(x,t)$ above, then the Fourier integral for $A(k)$ would give a delta function, corresponding to only one wave number for the wave.

An example of a wave $f(x,t)$ that is finite in extent is shown in Fig. 1a. The wave form shown is called a **wave packet**. The oscillations in the wave packet have a varying amplitude, called the **envelope** of the wave packet, that is generally a smooth function of limited extent. The particular wave packet in Fig. 1a has a Gaussian envelope, shown as the dashed curve in the figure. The initial form of the wave packet is

$$f(x,0) = g(x)e^{ik_0x} = e^{-x^2/2L^2}e^{ik_0x}, \tag{4}$$

where $g(x)$ is the envelope function and e^{ik_0x} is the oscillating function within the envelope. We have plotted the real part of this wave packet. The wave packet has the nominal wave number k_0, but the Fourier transform of $f(x,0)$ in Fig. 11.6b shows that there is actually a spread of wave numbers in the wave packet, described by the function

$$A(k) = \frac{L}{\sqrt{2\pi}}e^{-L^2(k-k_0)^2/2}. \tag{5}$$

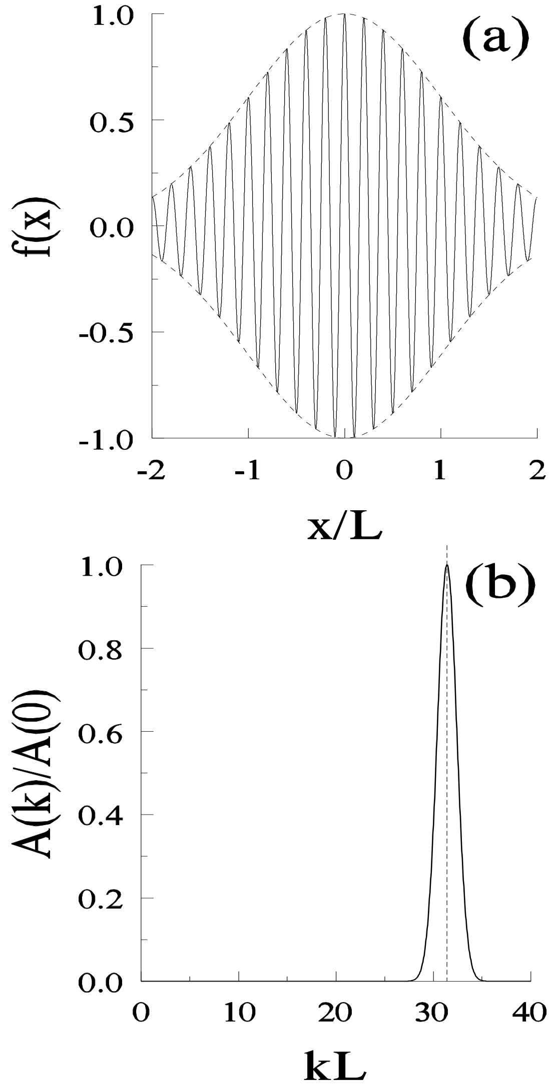

Fig. 1: (a) Wave packet, $f(x)$, with a Gaussian envelope, shown by the dashed curves. The width of the packet is given by $L = 10\lambda$.
(b) Fourier transform, $A(k)$, of the wave packet in (a). The central value k_0 is marked by the vertical dashed line.