The collapse of a quantum state as a joint probability construction

Peter Morgan

Yale University
peter.w.morgan@yale.edu

November 2nd, 2022

The Hitchhiker's Advanced Guide to Quantum Collapse Models

An FQXi and Templeton Foundation supported Workshop at INFN, Frascati (by Zoom)

outline

"The collapse of a quantum state as a joint probability construction",
PM, JPhysA 2022, https://doi.org/10.1088/1751-8121/ac6f2f

1. Classical mechanics is incomplete
2. Algebraic measurement theory for both CM and QM
3. Construct a connection between CM and QM that is not quantization,
   in terms of that algebraic measurement theory
4. How we use noncommutativity:
   (i) Measurement incompatibility
   (ii) "collapse" as joint measurement

[1] classical mechanics is incomplete

The usual idea:

QM Is Incomplete Because It Does Not Make Contact With CM

EPR 1935: "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?"

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Turn that around:

CM Is Incomplete Because It Does Not Make Contact With QM

2022: "Can Classical-Mechanical Description of Physical Reality Be Considered Complete?"

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Gleason, Kochen-Specker, Bell, ...

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various no-go theorems *show* that CM is *not* able to model measurements that *can* be modeled by QM

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but CM has been straw-manned

We can easily add two things to CM to make it as capable as QM:

noncommutativity and "quantum" noise

where "quantum" noise is different from "thermal" noise

[2] algebraic QM and CM — $\begin{smallmatrix}\text{noncommutative}\\ \text{or commutative}\end{smallmatrix}$ probability theory

There are abstract *measurements* $\hat{M}_1,\hat{M}_2,\hat{M}_3,\ldots,\hat{M}_1+\hat{M}_2,\ldots,\hat{M}_1\hat{M}_2,\ldots$

linear operators≡random variables, spectrum≡sample space, $\begin{smallmatrix}\text{noncommutative}\\ \text{or commutative}\end{smallmatrix}$, $\begin{smallmatrix}\text{associative,}\\ \text{distributive,}\\ \text{with unit}\end{smallmatrix}$

With no dynamics, the tradition is: QM=noncommutative, CM=commutative

A (statistical) *state* ρ maps measurement operators to *expected measurement results*

$$\rho(\hat{M}_1),\rho(\hat{M}_2),\rho(\hat{M}_3),\ldots,\rho(\hat{M}_1+\hat{M}_2),\ldots,\rho(\hat{M}_1\hat{M}_2),\ldots,\quad\rho(\hat{M}_1^n),\ldots$$

positive: $\rho(\hat{A}^\dagger\hat{A})\geq 0$; normalized: $\rho(1)=1$;

von Neumann linearity: $\rho(\lambda\hat{A}+\mu\hat{B})=\lambda\rho(\hat{A})+\mu\rho(\hat{B})$

compatible with the adjoint: $\rho(\hat{A}^\dagger)=\rho(\hat{A})^*$; where $(\hat{A}\hat{B})^\dagger=\hat{B}^\dagger\hat{A}^\dagger$

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We can also use measurement operators to *modulate* the state ρ to give different expected measurement results, $\rho_A(\hat{M}) = \frac{\rho(\hat{A}^\dagger\,\hat{M}\,\hat{A})}{\rho(\hat{A}^\dagger\,\hat{A})}$,

from which the GNS-construction gives us a Hilbert space

This has so far introduced neither Planck's constant nor a dynamics:

this is algebraic measurement theory in the abstract

Take classical mechanics to be an algebra of functions on phase space
that has *three* binary operations:
addition, multiplication, *and* the Poisson bracket

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a generator of time evolution, $\hat{Z}_H(u)=\{H,u\}$, the *Liouvillian*,
and we can use other functions v to give other transformations $\hat{Z}_v(u)$
$[\hat{Z}_v,\hat{Z}_w]=\hat{Z}_{\{v,w\}}\neq 0$ generates a noncommutative algebra

For $\hat{Y}_w(u)=w\cdot u$, $[\hat{Y}_v,\hat{Y}_w]=0$, but $[\hat{Z}_v,\hat{Y}_w]=\hat{Y}_{\{v,w\}}\neq 0$, generating
a noncommutative algebra of operators with addition and composition

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I suggest:
We can use the $\hat{Y}$'s *and* $\hat{Z}$*'s* of a more powerful CM_+ without restriction

Instead of quantization and its not-inverses
(the Correspondence Principle, the Ehrenfest theorem, decoherence, *et cetera*)
we can use the same measurement theory for CM_+ and QM

the classical simple harmonic oscillator

We work with the transformations generated by the Poisson bracket, not with the Poisson bracket directly
$\{u, v\} \not\Rightarrow [\hat{u}, \hat{v}]$

The Poisson bracket: $\{u, v\} = \frac{\partial u}{\partial p}\frac{\partial v}{\partial q} - \frac{\partial u}{\partial q}\frac{\partial v}{\partial p}$

$$\hat{Y}_q[u] = q \cdot u, \quad \hat{Z}_p[u] = \{p, u\} = \ \frac{\partial}{\partial q} u, \quad [\hat{Y}_q, \hat{Z}_p] = -1$$

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The Gibbs thermal state at temperature kT (in a generating function form, introducing j):

$$\langle_{\mathsf{kT}}|\mathrm{e}^{\mathrm{j}\lambda\hat{Y}_q+\mathrm{j}\mu\hat{Y}_p}|_{\mathsf{kT}}\rangle=\mathrm{e}^{-\mathsf{kT}(\lambda^2+\mu^2)/2},\quad \langle_{\mathsf{kT}}|\mathrm{e}^{\alpha\hat{Z}_p+\beta\hat{Z}_q}|_{\mathsf{kT}}\rangle=\mathrm{e}^{-(\alpha^2+\beta^2)/8\mathsf{kT}}$$

set $\hat{Y}_q=(a+a^\dagger)\sqrt{\mathsf{kT}}$, $\hat{Z}_p=\dfrac{(a-a^\dagger)}{2\sqrt{\mathsf{kT}}}$, $[a,a^\dagger]=1$, ensuring $[\hat{Y}_q,\hat{Z}_p]=-1$, and we set $a|_{\mathsf{kT}}\rangle=0$

We can construct modulated, non-equilibrium states, $\dfrac{\langle_{\mathsf{kT}}|\hat{A}^\dagger\,\hat{M}\,\hat{A}|_{\mathsf{kT}}\rangle}{\langle_{\mathsf{kT}}|\hat{A}^\dagger\hat{A}|_{\mathsf{kT}}\rangle}$, and hence a Hilbert space

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Instead of trying to map $(q,p) \not\Leftrightarrow (\hat{q},\hat{p})$, as quantization tries to *(but fails)*,

we *can* map CM_+ to QM, $(q,\mathrm{j}\frac{\partial}{\partial q}) \mapsto (\hat{q},\hat{p})$, $(p,\mathrm{j}\frac{\partial}{\partial p}) \mapsto (\hat{q}',\hat{p}')$

Crucially, kT is not $\hbar$, but it is also about an *irreducible* noise

quantum and thermal noise

What is the difference between quantum and thermal noise?

- $\hbar$ has units action, whereas kT has units energy
- In QFT, the quantum vacuum is Poincaré invariant, thermal noise is not
  This difference of symmetry properties *can* be used in CM_+
- In CM_+, $\hbar$ is an amplitude of Poincaré invariant noise
  kT is an amplitude of thermal noise

This gives a new reason to think that we must work with field theories,
because we can only define the Lorentz group in $1+n$-dimensions

For the Gibbs state of the Simple Harmonic Oscillator,
$\hat{Z}_H$ is anti-Hermitian, so we consider $\mathrm{j}\hat{Z}_H$, which is Hermitian,

$$\begin{aligned}\mathrm{j}\hat{Z}_H &= \mathrm{j}\left(p\cdot\frac{\partial}{\partial q} - q\cdot\frac{\partial}{\partial p}\right) = \mathrm{j}\left(\hat{Y}_p\hat{Z}_p + \hat{Y}_q\hat{Z}_q\right)\\ &= \mathrm{j}\left(ba - b^\dagger a^\dagger\right)\\ &= \tfrac{1}{2}\left[(a-\mathrm{j}b^\dagger)^\dagger(a-\mathrm{j}b^\dagger) - (a+\mathrm{j}b^\dagger)^\dagger(a+\mathrm{j}b^\dagger)\right] \not\geq 0\end{aligned}$$

$\boxed{\text{so } \langle_{\mathrm{kT}}|\mathrm{j}\hat{Z}_H|_{\mathrm{kT}}\rangle = 0}$

unboundedness of the Hermitian generators of time-like evolution

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\end{aligned}
$$

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The Hamiltonian operator in QM is bounded below$\to$analytic properties;

the corresponding operator in CM_+, $\mathrm{j}\hat{Z}_H$, is not (though $\hat{Y}_H$ is)

CM_+ includes (1) noncommutativity and (2) quantum noise, however

(3) analyticity is mathematically useful but is *not* included

We can think of QM as an analytic form of CM_+

Accepting this instead of trying to fix it gives us isomorphisms,

as we have seen, which is pleasantly different from quantization

reprise: classical and quantum measurement theories

If "quantum" noise pushes us to field theory, what is the role of particles?

- For your consideration: QM and QFT are formalisms about
  Megabytes or Terabytes of experimental records of *events*
- but assigning events to particles, against a noisy background,
  will generally be a *fragile* algorithm
  We have to consider patterns of events *globally*

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- *but* assigning events to particles, against a noisy background,
  will generally be a *fragile* algorithm
  We have to consider patterns of events *globally*
- For an empiricist, QM is not enough about particles and systems
  for particle properties to be hard-wired into QM's axioms
- Particles are *not* hard-wired into QFT's axioms
  and nor should they be for classical noisy fields
- This focus connects with Bohr's insistence on classical description
  "It is decisive to recognize that, however far the phenomena transcend the scope of classical physical explanation, the account of all evidence must be expressed in classical terms."
  but "in classical terms" about *events*, not about particles and their properties

[4] Two sides of noncommutativity:

(i) measurement incompatibility for CM_+

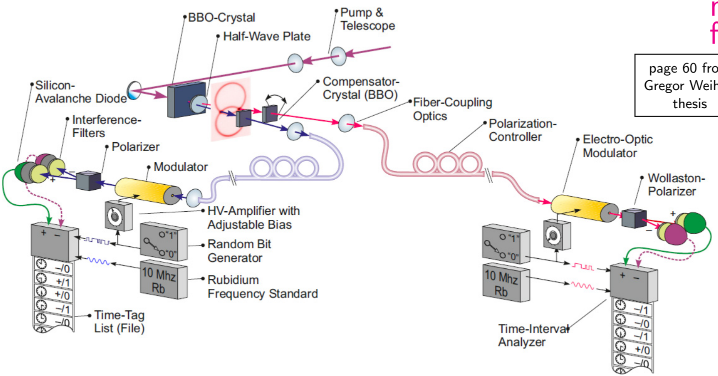
(ii) "collapse" as joint measurement for QM

[4(i)] measurement incompatibility in practice

Alice and Bob both have two Avalanche PhotoDiodes,
an Electro-Optic Modulator, a Random Bit Generator, and a clock;
a central apparatus modulates the ground state

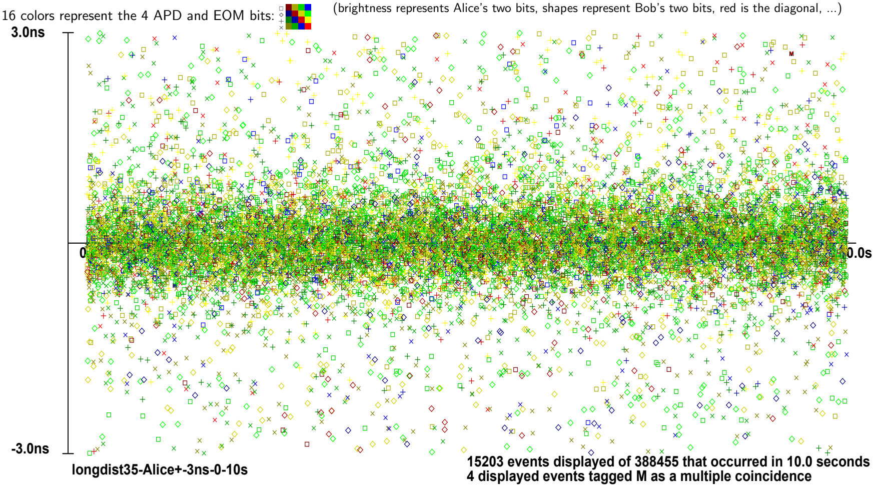
think noisy fields

page 60 from Gregor Weihs's thesis



The time when an APD's signal rises to a higher level is recorded, and
which APD it was, and what the EOM setting was: when and 2 bits
This compressed record does not analyze any other signal details

Gregor gets measurement results (Alice sees almost 400,000 APD events in 10 seconds)

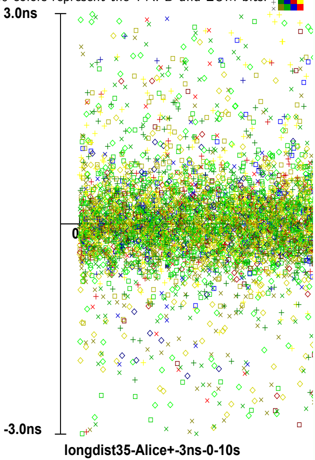


For over 15,000 of Alice's 400,000 events, Bob also records an event within 3 nanoseconds

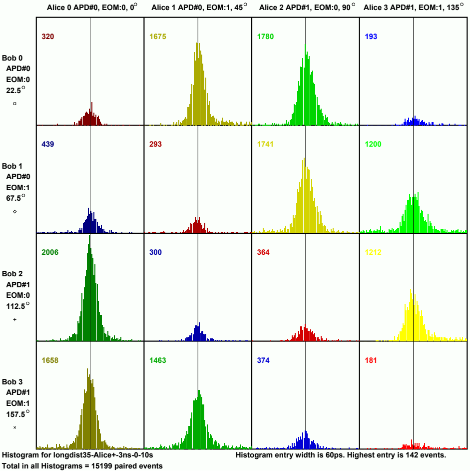
When Alice and Bob both record an event within 3 nanoseconds, the majority are green or yellow

Gregor gets measurement results (Alice sees almost 400,000 APD events in 10 seconds)

16 colors represent the 4 APD and EOM bits:



(brightness represents Alice's two bits, shapes represent Bob's two bits, red is the diagonal, ...)



E00 = -0.694 [320+364,-2006-1780]
E01 = -0.614 [439+374,-1658-1741]
E10 = 0.708 [1675+1212,-300-193]
E11 = -0.698 [293+181,-1463-1200]

|E00-E10|+|E01+E11| = 2.714
|E00-E01|+|E10+E11| = 0.090

after simultaneous events have been identified, absolute timing information is discarded

then relative timing information is also discarded

to give a 4×4 table of APD# and EOM setting, 2 bits for Alice and 2 bits for Bob

transformations and noncommutativity

We have applied various transformations to recorded experimental data
If they were innocuous, we could use commutative algebras
as models of those transformations

In QM, we model Bell-violating statistics using noncommuting operators

In CM as usual, we do not have noncommuting operators

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In Koopman's Hilbert space formalism for classical mechanics, CM_+, we can use noncommutativity as needed to model contexts systematically

- For quantum fields, locality is *closely* associated with incompatibility because *microcausality* only allows noncommutativity at time-like separation

The 4×4 table of numbers we constructed could come from anywhere,
so, for now, set aside discussion of locality (and this talk *is* about probability)

Boole 1854

It has been known since George Boole in the mid-19th Century that
for *some* pairs of probability measures we *cannot* construct
a joint probability measure that has that pair
as marginal probability measures

It is classically understandable that such pairs can arise when
measurement results come from different experimental contexts

"Measurement incompatibility" is classically understandable and
classical mechanics should have a systematic response to it
so we can optimize our use of the results of new experiments

We can use Wigner functions in CM_+ just as we do in QM

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For a measurement A, with sample space $\mathcal{A} = \{a_m\}$, $\hat{A} = \sum_m a_m \hat{P}_m$, and
a measurement B, with sample space $\mathcal{B} = \{b_n\}$, $\hat{B} = \sum_n b_n \hat{Q}_n$,

For solo measurements, with density operator $\hat{\rho}$,
we obtain the result α_m with probability $\mathrm{Tr}[\hat{\rho}\hat{P}_m]$ and
we obtain the result β_n with probability $\mathrm{Tr}[\hat{\rho}\hat{Q}_n]$.

For two measurements, of A first, followed by B, we say that the result α_m "collapses" the state from $\hat{\rho}$ to the collapsed state $\hat{\rho}_m$,

$$\hat{\rho}_m = \frac{\hat{P}_m\hat{\rho}\hat{P}_m}{\mathrm{Tr}[\hat{P}_m\hat{\rho}\hat{P}_m]} = \frac{\hat{P}_m\hat{\rho}\hat{P}_m}{\mathrm{Tr}[\hat{\rho}\hat{P}_m]},$$

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then we measure B in that state, so we obtain the result α_m followed by β_n with conditional probability

$$p(\beta_n|\alpha_m) = \mathsf{Tr}[\hat{\rho}_m\hat{Q}_n] = \frac{\mathsf{Tr}[\hat{P}_m\hat{\rho}\hat{P}_m\hat{Q}_n]}{\mathsf{Tr}[\hat{\rho}\hat{P}_m]}.$$

The joint probability, therefore, is

$$p(\alpha_m \text{ and } \beta_n) = \mathsf{Tr}[\hat{P}_m\hat{\rho}\hat{P}_m \cdot \hat{Q}_n] = \mathsf{Tr}[\hat{\rho} \cdot \hat{P}_m\hat{Q}_n\hat{P}_m].$$

We have $p(\alpha_m \text{ and } \beta_n) = \mathrm{Tr}[\hat{P}_m\hat{\rho}\hat{P}_m \cdot \hat{Q}_n] = \mathrm{Tr}[\hat{\rho} \cdot \hat{P}_m\hat{Q}_n\hat{P}_m]$,

so the positive operators $\hat{J}_{mn} = \hat{P}_m\hat{Q}_n\hat{P}_m$ generate

the joint probabilities $\mathrm{Tr}[\hat{\rho}\hat{J}_{mn}]$.

Instead of collapse affecting a state,

we can take collapse to affect the next measurement

If $[\hat{A},\hat{B}]=0$, then $\hat{P}_m\hat{Q}_n\hat{P}_m = \hat{P}_m\hat{Q}_n = \hat{Q}_n\hat{P}_m\hat{Q}_n \sim$ no action

We can use $\hat{J}_{mn}$ to construct a "collapse product",

a measurement A▸◦B, with sample space $\mathcal{A}\times\mathcal{B}$, even if $[\hat{A},\hat{B}]\neq 0$

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The existence of a joint probability is traditionally "classical", so we can instead use
$$\mathsf{Tr}[\hat{\rho}'\cdot\hat{P}'_m\hat{Q}'_n]=\mathsf{Tr}[\hat{\rho}\cdot\hat{P}_m\hat{Q}_n\hat{P}_m], \qquad \text{with } [\hat{A}',\hat{B}']=0,\ \hat{\rho}'\neq\hat{\rho}$$

We can think of this as a *"super-Heisenberg picture"*, for which
unitary evolution *and* collapse are *both* applied to measurements
or as the *"Bohr picture"*, because it is rather classical

we can (and somehow must) extend this to many measurements

For a sequence of three or more measurements,
we can use the *sequential product*, $\hat{X} \circ \hat{Y} = \sqrt{\hat{X}} \cdot \hat{Y} \cdot \sqrt{\hat{X}}$
(or more elaborate constructions of positive operators)

Collapse of the quantum state after measurement is ambiguous

$$\rho\Big(\sqrt{\hat{P}_i^{(A)}\hat{P}_j^{(B)}\hat{P}_i^{(A)}}\,\hat{P}_k^{(C)}\sqrt{\hat{P}_i^{(A)}\hat{P}_j^{(B)}\hat{P}_i^{(A)}}\Big) \text{ or } \rho\big(\hat{P}_i^{(A)}\hat{P}_j^{(B)}\hat{P}_k^{(C)}\hat{P}_j^{(B)}\hat{P}_i^{(A)}\big)?$$

$$(\mathrm{A}\blacktriangleright\!\!\circ\mathrm{B})\blacktriangleright\!\!\circ\mathrm{C} \qquad \neq \qquad \mathrm{A}\blacktriangleright\!\!\circ(\mathrm{B}\blacktriangleright\!\!\circ\mathrm{C})$$

We can use any ordering, but each makes a different assertion about collapses

This is nonassociative, so, more complicated than the Heisenberg cut,
we have a *Heisenberg ordering ambiguity*

we can (and somehow must) extend this to many measurements

For a sequence of three or more measurements,
we can use the *sequential product*, $\hat{X} \circ \hat{Y} = \sqrt{\hat{X}} \cdot \hat{Y} \cdot \sqrt{\hat{X}}$
(or more elaborate constructions of positive operators)

Collapse of the quantum state after measurement is ambiguous

$$\rho\Big(\sqrt{\hat{P}^{(A)}_i \hat{P}^{(B)}_j \hat{P}^{(A)}_i}\, \hat{P}^{(C)}_k \sqrt{\hat{P}^{(A)}_i \hat{P}^{(B)}_j \hat{P}^{(A)}_i}\Big) \text{ or } \rho\big(\hat{P}^{(A)}_i \hat{P}^{(B)}_j \hat{P}^{(C)}_k \hat{P}^{(B)}_j \hat{P}^{(A)}_i\big)?$$

$$(\mathrm{A} \blacktriangleright\!\!\circ \mathrm{B}) \blacktriangleright\!\!\circ \mathrm{C} \qquad \neq \qquad \mathrm{A} \blacktriangleright\!\!\circ (\mathrm{B} \blacktriangleright\!\!\circ \mathrm{C})$$

We can use any ordering, but each makes a different assertion about collapses

This is nonassociative, so, more complicated than the Heisenberg cut,
we have a *Heisenberg ordering ambiguity*

For signal analysis, when we have *many* measurements at time-like separation,
we can use $\mathrm{A}_1, \ldots, \mathrm{A}_{100\ldots000}$ with many ambiguous collapses,
or we can use $\mathrm{A}'_1, \ldots, \mathrm{A}'_{100\ldots000}$, which all commute, unambiguously
which we can think of as Bohr's ideal of a classical model for compatible measurements
measurements at timelike separation *can* give joint probabilities

"Collapse" is not

⟶only or necessarily⟵
a dynamical process

We can ⟶also⟵ take it to be a
JOINT PROBABILITY
ALGORITHM

Belavkin(1994) Quantum Non-Demolition (QND) Measurements
Tsang&Caves(2012) Quantum-Mechanics–Free-Subsystems

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An unfortunate but necessary tradeoff:

QM is effective for incompatible measurements, but less so for joint measurements

Collapse is QM's way of constructing joint measurement probabilities

CM is effective for joint measurements, but less so for incompatible measurements

The Poisson bracket is CM_+'s way of constructing incompatible measurements

events

An event in an APD compared with an event from throwing a coin:

We throw a coin, we see it land, we record '0' or '1'

We 'throw' an APD, we record the 'signal' as '0' or '1', at GHz rates

We have engineered the avalanche thermodynamics of the APD so it is like a coin for the purposes of statistics: the 'signal' is either '0' or '1'

The difference: there are many '0's together and many '1's together, so we can *compress* the data by recording only the times of transitions

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Throws of a coin or of an APD are both used to compute relative frequencies, with the dynamics effectively abstracted away

That we *can* work with *generalized* probability models for throws of a coin or of an APD does not deny interest in *also* working with *thermo-*dynamical "collapse" models

classical and quantum models

With noncommutativity and “quantum” noise added into CM_+,
we can allow ourselves, with care, to think classically

We should be systematic about contextuality and Boole's incompatibility
as classically natural ideas

Noncommutativity lets us use information *systematically* that otherwise
we might have to discard as not relevant to a new experiment

Insofar as experiments can be described using CM_+ or QM,
quantum systems can be thought of as $\mathrm{classical}_+$ systems (and vice versa)

Quantum mechanics can be thought of as a generalized probability theory,
for which the generalization is classically understandable

The difference between Quantum and Classical is subtle, but not mysterious

if we think in terms of events and CM_+, *not* in terms of particle and system properties

An experiment behaves the same whether we use CM_+ or QM models
We might, however, choose to construct different experiments

Instead of collapse of the state, we can use Quantum Non-Demolition measurements, but we still need noncommutativity to model contextuality/incompatibility

Quantum and Classical have been converging, in numerous ways, for decades

Generalized Probability Theories, phase space methods, contextuality, non-demolition measurement, Koopman CM, time-frequency analysis, stochastic methods, semi-classical methods, (superdeterminism)

We will, however, continue to use quantum mechanics $\boxed{\text{for its analyticity}}$

"Classical states, quantum field measurement", Physica Scripta 2019
"An algebraic approach to Koopman classical mechanics", Annals of Physics 2020
"The collapse of a quantum state as a joint probability construction", Journal of Physics A 2022

and, ancient history, "Bell inequalities for random fields", Journal of Physics A 2006