DERIVING $\Gamma^{j'}{}_{0'0'}$

So far I have established that

$$R^{1'}{}_{0'1'0'} = -\frac{2GM}{R^3} \quad \& \quad R^{2'}{}_{0'2'0'} = \frac{GM}{R^3}$$

$$(R^{1'}{}_{0'2'0'} = 0).$$

These results align with the analysis in MTW chapter 1, p29.

There are two separate sets of equations for geodesic motion in this particular system, ie

$$\frac{d^2x^\alpha}{dt^2} + R^\alpha{}_{0\alpha 0}x^\alpha = 0 \quad \text{and} \quad \frac{d^2x^\alpha}{dt^2} + \Gamma^\alpha{}_{00} = 0$$

So I need to show that $R^{1'}{}_{0'1'0'}\,x' = \Gamma^{1'}{}_{0'0'}$

and $R^{2'}{}_{0'2'0'} = \Gamma^{2'}{}_{0'0'}$ by deriving $\Gamma^{j'}{}_{0'0'}$ using

equation (12.14) $\Rightarrow \Gamma^{j'}{}_{0'0'} = \dfrac{\partial \Phi}{\partial x^{j'}} + A_{j'k}\left(\ddot{A}_{l'k}\,x^{l'} - \ddot{a}^k\right)$.

The values of $A_{j'k}$, a^k etc are obtained from the original transformation ie

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \overset{A_{j'k}}{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -R \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + \overset{a^j}{\begin{pmatrix} R\cos\theta \\ R\sin\theta \end{pmatrix}}$$

Now $\Phi = \dfrac{-GM}{(x^2+y^2)^{1/2}}$

$$= \frac{-GM}{\left[(R\cos\theta + x'\cos\theta - y'\sin\theta)^2 + (R\sin\theta + x'\sin\theta + y'\cos\theta)^2\right]^{1/2}}$$

$$= \frac{-GM}{\left[\begin{pmatrix} R^2\cos^2\theta + x'^2\cos^2\theta + y'^2\sin^2\theta \\ +2Rx'\cos^2\theta - 2Ry'\cos\theta\sin\theta - 2x'y'\cos\theta\sin\theta \end{pmatrix} + \begin{pmatrix} R^2\sin^2\theta + x'^2\sin^2\theta + y'^2\cos^2\theta \\ +2Rx'\sin^2\theta + 2Ry'\cos\theta\sin\theta + 2x'y'\sin\theta\cos\theta \end{pmatrix}\right]^{1/2}}$$

$$= \frac{-GM}{(R^2 + x'^2 + y'^2 + 2Rx')^{1/2}}.$$

$$\frac{\partial\Phi}{\partial x'} = \frac{GMx'}{(R^2+x'^2+y'^2+2Rx')^{3/2}} + \frac{GMR}{(R^2+x'^2+y'^2+2Rx')^{3/2}}.$$

$$= \frac{GMx'}{R^3\left(1+\frac{2x'}{R}\right)^{3/2}} + \frac{GMR}{R^3\left(1+\frac{2x'}{R}\right)^{3/2}} \quad \left[\frac{x'^2}{R^2} \ll \frac{x'}{R}\right]$$

$$= \frac{GMx'}{R^3}\left(1 - \frac{3x'}{R}\right) + \frac{GMR}{R^3}\left(1 - \frac{3x'}{R}\right)$$

$$= \frac{GMx'}{R^3} - \frac{3GMx'^2}{R^4} + \frac{GM}{R^2} - \frac{3GMx'}{R^3}$$

$$= -\frac{2GMx'}{R^3} + \frac{GM}{R^2} \quad \left[\frac{x'}{R} \ll 1\right]$$

$$\frac{\partial\Phi}{\partial y'} = \frac{GMy'}{(R^2+x'^2+y'^2+2Rx')^{3/2}} = \frac{GMy'}{R^3\left(1+\frac{2x'}{R}\right)^{3/2}} = \frac{GMy'}{R^3}$$

We can now proceed to the evaluations of $\Gamma^{j'}{}_{0'0'}$ using

$$\Gamma^{j'}{}_{0'0'} = \frac{\partial \Phi}{\partial x'} + A_{j'k}\left(\ddot{A}_{l'k}x^{l'} - \ddot{a}^{k}\right)$$

where $\quad a^{k} = A_{j'k}a^{j'}$

$$a^{1} = A_{1'1}a^{1'} + A_{2'1}a^{2'} \qquad a^{2} = A_{1'2}a^{1'} + A_{2'2}a^{2'}$$

$$= -R\cos\theta \qquad\qquad = -R\sin\theta$$

$$\ddot{a}^{1} = R\cos\theta\,\omega^{2} \qquad a^{2} = R\sin\theta\,\omega^{2} \quad \left[\omega = \frac{d\theta}{dt} \quad \frac{d\omega}{dt} = 0\right]$$

$$\Gamma^{1'}{}_{0'0'} = \frac{\partial\Phi}{\partial x'} + A_{1'k}\left(\ddot{A}_{l'k}x^{l'} - \ddot{a}^{k}\right)$$

$$= \frac{\partial\Phi}{\partial x'} + A_{1'1}\left(\ddot{A}_{1'1}x' + A_{2'1}y' - \ddot{a}^{1}\right) + A_{1'2}\left(A_{1'2}x' + A_{2'2}y' - \ddot{a}^{2}\right)$$

$$= \frac{\partial\Phi}{\partial x'} + \cos\theta\left(-\cos\theta\,\omega^{2}x' + \sin\theta\,\omega^{2}y' - R\omega^{2}\cos\theta\right) + \sin\theta\left(-\sin\theta\,\omega^{2}x' - \cos\theta\,\omega^{2}y' - R\omega^{2}\sin\theta\right)$$

$$= \frac{\partial\Phi}{\partial x'} - \omega^{2}x' - R\omega^{2}.$$

$$= -\frac{2GM}{R^{3}}x' + \frac{GM}{R^{2}} - \omega^{2}x' - R\omega^{2}$$

$$= -\frac{3GM}{R^{3}}x' \qquad \left(\omega^{2} = \frac{GM}{R^{3}}\right)$$

$$\Gamma^{2'}{}_{0'0'} = \frac{\partial\Phi}{\partial y'} + (-\sin\theta)\left(-\cos\theta\,\omega^{2}x' + \sin\theta\,\omega^{2}y' - R\omega^{2}\cos\theta\right) + (\cos\theta)\left(-\sin\theta\,\omega^{2}x' - \cos\theta\,\omega^{2}y' - R\omega^{2}\sin\theta\right)$$

$$= \frac{\partial\Phi}{\partial y'} - \omega^{2}y'$$

$$= \frac{GMy'}{R^{3}} - \omega^{2}y' \quad = 0$$

So I've ended up with:

$$\Gamma^{1'}_{0'0'} = -\frac{3GM x'}{R^3} \quad \text{instead of} \quad -\frac{2GM x'}{R^3}$$

and

$$\Gamma^{2'}_{0'0'} = 0 \quad \text{instead of} \quad \frac{GM}{R^3} y'.$$

Can you see where I am going wrong?

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