rotations. The matrix corresponding to a rotation about the z axis through an angle $\theta = N\delta\theta$ $(N \to \infty)$ is clearly

$$\begin{aligned} R_z(\theta) &= [R_z(\delta\theta)]^N \\ &= (1 + iJ_z\delta\theta)^N \\ &= \left(1 + iJ_z\frac{\theta}{N}\right)^N \\ &= e^{iJ_z\theta}. \end{aligned} \tag{2.34}$$

We may check that this yields the required matrix (2.28). Defining the exponential by its power series expansion, we have

$$\begin{aligned} e^{iJ_z\theta} &= 1 + iJ_z\theta - J_z^2\frac{\theta^2}{2!} - iJ_z^3\frac{\theta^3}{3!} + \cdots \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \theta\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{\theta^2}{2!}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &\quad + \frac{\theta^3}{3!}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \cdots \\ &= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

which is (2.28). A finite rotation about an axis $\mathbf{n}$ through an angle θ is denoted

$$R_n(\theta) = e^{i\mathbf{J}\cdot\boldsymbol{\theta}} = e^{i\mathbf{J}\cdot\mathbf{n}\theta} \tag{2.35}$$

where $\boldsymbol{\theta} = \mathbf{n}\theta$.

So much for the rotation group. Now consider the group $SU(2)$, consisting of 2×2 unitary matrices with unit determinant

$$UU^\dagger = 1, \det U = 1. \tag{2.36}$$

Putting

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.37}$$

the unitarity condition reads $U^+ = U^{-1}$, which, since $\det U = 1$, becomes

$$\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

and hence $a^* = d$, $b^* = -c$. Then $\det U = |a|^2 + |b|^2$, so we have

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \quad |a|^2 + |b|^2 = 1. \tag{2.38}$$

This is regarded as the transformation matrix in a 2-dimensional complex space with basic spinor $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$;

$$\xi \to U\xi, \quad \xi^\dagger \to \xi^\dagger U^\dagger. \tag{2.39}$$

It is clear that

$$\xi^\dagger \xi = |\xi_1|^2 + |\xi_2|^2$$

is invariant under (2.39). On the other hand, the outer product

$$\xi\xi^\dagger = \begin{pmatrix} |\xi_1|^2 & \xi_1\xi_2^* \\ \xi_2\xi_1^* & |\xi_2|^2 \end{pmatrix} \to U\xi\xi^\dagger U^\dagger. \tag{2.40}$$

Observe that $\xi\xi^\dagger$ is a Hermitian matrix.

We see from (2.39) that ξ and $\xi^\dagger$ transform in different ways, but we may use the unitarity of U to show that $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ and $\begin{pmatrix} -\xi_2^* \\ \xi_1^* \end{pmatrix}$ transform in the same way under $SU(2)$. We have, comparing (2.38) and (2.39),

$$\left.\begin{aligned} \xi_1' &= a\xi_1 + b\xi_2, \\ \xi_2' &= -b^*\xi_1 + a^*\xi_2, \end{aligned}\right\} \tag{2.41}$$

and hence

$$\left.\begin{aligned} -\xi_2^{*\prime} &= a(-\xi_2^*) + b\xi_1^*, \\ \xi_1^{*\prime} &= -b^*(-\xi_2^*) + a^*\xi_1^*. \end{aligned}\right\} \tag{2.42}$$

Now

$$\begin{pmatrix} -\xi_2^* \\ \xi_1^* \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_1^* \\ \xi_2^* \end{pmatrix} = \zeta\xi^* \tag{2.43}$$

where

$$\zeta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.44}$$

so we have shown that ξ and $\zeta\xi^*$ transform in the same way under $SU(2)$; in symbols

$$\xi \sim \zeta\xi^* \tag{2.45}$$

('$\sim$' means 'transforms like'). Hence

$$\xi^\dagger \sim (\zeta\xi)^{\mathrm{T}} = (-\xi_2, \xi_1), \tag{2.46}$$

and

$$\xi\xi^\dagger \sim \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} (-\xi_2, \xi_1) = \begin{pmatrix} -\xi_1\xi_2 & \xi_1^2 \\ -\xi_2^2 & \xi_1\xi_2 \end{pmatrix}. \tag{2.47}$$

Calling this matrix $-H$, we see from (2.40) that under an $SU(2)$ transformation

$$H \to UHU^\dagger; \tag{2.48}$$

in addition, H is a traceless matrix.

We may now construct, from the position vector $\mathbf{r}$, a traceless 2×2 matrix transforming under $SU(2)$ like H. It is

$$h = \boldsymbol{\sigma} \cdot \mathbf{r} = \begin{pmatrix} z & x - \mathrm{i}y \\ x + \mathrm{i}y & -z \end{pmatrix} \tag{2.49}$$

where the matrices $\boldsymbol{\sigma}$ are the well-known Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \tag{2.50}$$

h is Hermitian, and the transformation

$$h \to UhU^\dagger = h' \tag{2.51}$$

preserves the Hermiticity and tracelessness of h if U is unitary. In addition, if U belongs to $SU(2)$, and so has determinant 1, then $\det h' = \det h$, or

$$x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2; \tag{2.52}$$

the unitary transformation (2.51) on h induces a *rotation* of the position vector $\mathbf{r}$. Identifying H and h, we conclude finally that

■ An $SU(2)$ transformation on $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \equiv O(3)$ transformation on $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

with

$$x = \tfrac{1}{2}(\xi_2^2 - \xi_1^2), \quad y = \frac{1}{2\mathrm{i}}(\xi_1^2 + \xi_2^2), \quad z = \xi_1 \xi_2. \tag{2.53}$$

The parameters of an $SU(2)$ transformation are a, b, both complex, with one condition: $|a|^2 + |b|^2 = 1$. There are thus three real parameters, just like there are for a rotation. We shall now find an explicit relation between the two sets of parameters. Taking the square and the product of the two relations (2.41), and using the expressions for x, y and z given above (equation (2.53)), we have, under $SU(2)$,

$$\begin{aligned} x' &= \tfrac{1}{2}(a^2 + a^{*2} - b^2 - b^{*2})x - \frac{\mathrm{i}}{2}(a^2 - a^{*2} + b^2 - b^{*2})y - (a^*b^* + ab)z, \\ y' &= \frac{\mathrm{i}}{2}(a^2 - a^{*2} - b^2 + b^{*2})x + \tfrac{1}{2}(a^2 + a^{*2} + b^2 + b^{*2})y - \mathrm{i}(ab - a^*b^*)z, \\ z' &= (ab^* + ba^*)x + (ba^* - ab^*)\mathrm{i}y + (|a|^2 - |b|^2)z. \end{aligned} \tag{2.54}$$

Now we put $a = \mathrm{e}^{\mathrm{i}\alpha/2}$, $b = 0$ (which obeys $|a|^2 + |b|^2 = 1$). Equation (2.54) gives

$$\begin{aligned} x' &= x \cos \alpha + y \sin \alpha, \\ y' &= -x \sin \alpha + y \cos \alpha, \\ z' &= z, \end{aligned}$$