SOLUTIONS FOR THE FINAL EXAM UCU SCI 211, DECEMBER 2002

1a) Since the initial velocity is zero and the wave propagation speed is $c = 1$, the d'Alembert formula takes the form $v(x,t) = \frac{1}{2}(f(x+t) + f(x-t))$; in particular, for $x = 0$ we have $v(0,t) = \frac{1}{2}(f(t) + f(-t)) = 0$ for every t (because f is an odd function, so $f(-t) = -f(t)$).

Remarks. Being a restriction to $x \geq 0$ of a solution $u(x,t)$ of the wave equation, $v(x,t)$ is itself a solution in the domain $x > 0$ (this is tautology: $\frac{\partial^2 v(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial t^2}$ and $\frac{\partial^2 v(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial x^2}$ for all positive x (because $v(x,t) = u(x,t)$ for $x > 0$), so $\frac{\partial^2 v(x,t)}{\partial t^2} = \frac{\partial^2 v(x,t)}{\partial x^2}$ (for all t and $0 < x < \infty$), because $\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2}$). The initial condition is satisfied tautologically, too: $v(x,0) = u(x,0) = f(x) = \varphi(x)$ for $x > 0$ (the first equality says that we use u as v when $x > 0$, the second equality holds because $f(x)$ is the initial condition for $u(x,t)$, the last one holds because f is a continuation of φ, which coincides with φ for $x > 0$). The boundary condition is $v(0,t) = 0$, and this is indeed so: $v(0,t) = u(0,t) = 0$ by virtue of d'Alembert's formula, as it is shown above. Finally, vanishing of the second derivative $\varphi''(0)$ (which is, strictly speaking, not defined at the endpoint of the domain of φ), mentioned in the problem, guarantees that the solution $u(x,t)$ is twice continuously differentiable everywhere.

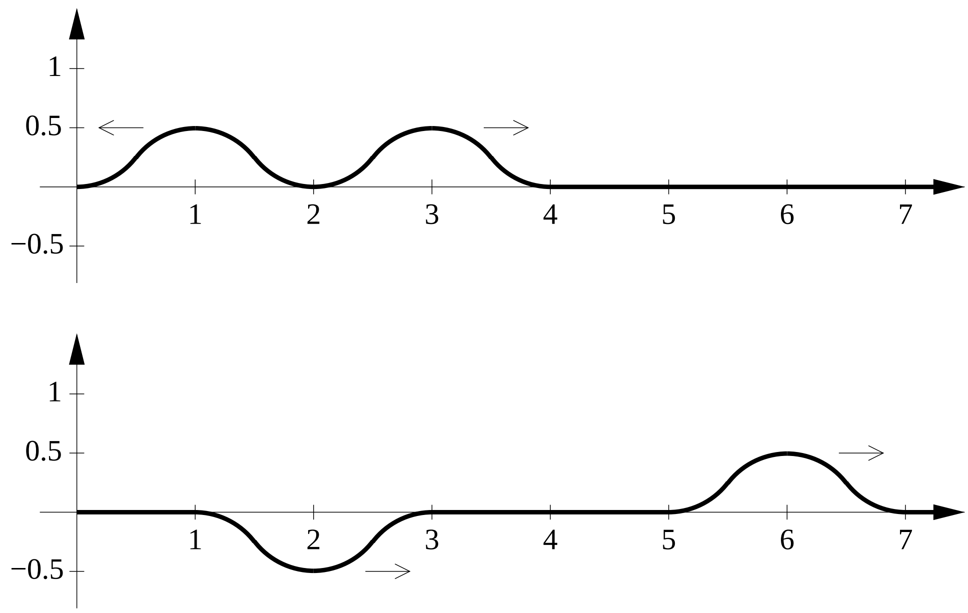

FIGURE 1. Graphs of $v(x,1)$ and $v(x,4)$

1b) See the pictures above.

Remark. The hump splits into two equal humps of height 1/2. One of them runs to the right with unit velocity; the other one first moves to the left, but bounces at the endpoint $x = 0$ and gets reverted. See the notebook partialDE.nb for more examples.

2a) The Fourier series for $f(t)$ is $\sum_{k=1}^{\infty} b_k \sin(k\pi t)$; there are sine terms only, because f is an odd function. The Fourier coefficients are $b_k = 2\int_0^1 f(t)\sin(k\pi t)\,dt =$

$\ldots = \frac{2}{k\pi}(1-(-1)^k)$, so $b_k = 0$ if k is even and $b_k = \frac{4}{k\pi}$ if k is odd. So we have

$$f(t) = \sum_{k\geq 1,\, k \text{ odd}} \frac{4}{k\pi}\sin(k\pi t)\,,$$

and this equals 1 for $0 < t < 1$ (see Theorem 1.1 in the Guide Book). The double infinite sum below does not converge absolutely, so we will treat it as the limit of finite sums:

$$\begin{aligned}
&\sum_{l\geq 1,\, l \text{ odd}} \sum_{k\geq 1,\, k \text{ odd}} \frac{4}{k\pi}\sin(k\pi x)\frac{4}{l\pi}\sin(l\pi y) \\
&\quad = \lim_{N\to\infty}\left(\sum_{l=1,\, l \text{ odd}}^{N} \sum_{k=1,\, k \text{ odd}}^{N} \frac{4}{k\pi}\sin(k\pi x)\frac{4}{l\pi}\sin(l\pi y)\right) \\
&\quad = \lim_{N\to\infty}\left(\sum_{k=1,\, k \text{ odd}}^{N} \frac{4}{k\pi}\sin(k\pi x)\right)\left(\sum_{l=1,\, l \text{ odd}}^{N} \frac{4}{l\pi}\sin(l\pi y)\right) \\
&\quad = \sum_{k\geq 1,\, k \text{ odd}} \frac{4}{k\pi}\sin(k\pi x) \sum_{l\geq 1,\, l \text{ odd}} \frac{4}{l\pi}\sin(l\pi y) \\
&\quad = f(x)f(y) = 1, \qquad 0 < x < 1, \quad 0 < y < 1.
\end{aligned}$$

This proves statement a).

2b) If x or y equals 0 or 1, then $\sin(k\pi x)\sin(l\pi y) = 0$, so all summands vanish on the boundary of the unit square. Thus $u(x,t)$ satisfies the boundary conditions. To compute the Laplacian of $u(x,y)$, we take the sum of Laplacians of the summands. Note that $\frac{\partial^2}{\partial x^2}\sin(k\pi x) = -k^2\pi^2\sin(k\pi x)$ and $\Delta(\sin(k\pi x)\sin(l\pi y)) = -(k^2+l^2)\pi^2\sin(k\pi x)\sin(l\pi y)$.

$$\begin{aligned}
\Delta u(x,y) &= \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} \\
&= -\sum_{l\geq 1,\, l \text{ odd}} \sum_{k\geq 1,\, k \text{ odd}} \Delta\left(\frac{16}{\pi^4 kl(k^2+l^2)}\sin(k\pi x)\sin(l\pi y)\right) \\
&= -\sum_{l\geq 1,\, l \text{ odd}} \sum_{k\geq 1,\, k \text{ odd}} \frac{16}{\pi^4 kl(k^2+l^2)}(-\pi^2(k^2+l^2)\sin(k\pi x)\sin(l\pi y)) \\
&= \sum_{l\geq 1,\, l \text{ odd}} \sum_{k\geq 1,\, k \text{ odd}} \frac{4}{k\pi}\frac{4}{l\pi}\sin(k\pi x)\sin(l\pi y)) = 1
\end{aligned}$$

inside the unit square, according to problem 2a).

3a) Euler step: $x_{n+1} = x_n + hcx_n$. Initial condition $x(0) = a$ yields base of induction: $x_0 = a = a(1+hc)^0$. Induction step: suppose we know that $x_n = a(1+hc)^n$ for all $n \leq k$. Then for $n = k+1$ we have $x_{k+1} = x_k + hcx_k = (1+hc)x_k = (1+hc)a(1+hc)^k = a(1+hc)^{k+1}$, so the formula $x_n = a(1+hc)^n$ holds for $n = k+1$, too.

3b) The solution of the differential equation $dx(t)/dt = cx(t)$ with the initial condition $x(0) = a$ is $x(t) = ae^{ct}$, so $\ln x(t) = ct + \ln a$. Combining $x_N = a(1+hc)^N$ with $h = t/N$, we get $\ln x_N(t) = \ln a + N \ln(1+ct/N)$, and then $v(t) = ct - N\ln(1+ct/N)$, which implies $v(0) = 0$ and $\frac{dv(t)}{dt} = c - N\frac{1}{1+ct/N}\frac{c}{N} = \frac{c^2t}{N+ct}$. If $0 \le t \le T$, then $N \le N + ct \le N + cT$, thus

$$\frac{c^2t}{N+cT} \le \frac{dv(t)}{dt} \le \frac{c^2t}{N}. \tag{1}$$

Finally, $v(t) = v(0) + \int_0^t v'(s)\,ds$. Taking into account that $v(0) = 0$, we deduce from (1) that $\frac{c^2}{N+cT}\int_0^t s\,ds \le v(t) \le \frac{c^2}{N}\int_0^t s\,ds$, or

$$\frac{c^2t^2}{2(N+cT)} \le v(t) \le \frac{c^2t^2}{2N}$$

for all $t \in [0, T]$, q.e.d.