Example. $L = \mathfrak{sl}(2, \mathsf{F})$, $V = \mathsf{F}^2$, ϕ the identity map $L \to \mathfrak{gl}(V)$. Let (x, h, y) be the standard basis of L (2.1). It is quickly seen that the dual basis relative to the trace form is $(y, h/2, x)$, so $c_\phi = xy+(1/2)h^2+yx = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix}$. Notice that $3/2 = \dim L/\dim V$.

When ϕ is no longer faithful, a slight modification is needed. Ker ϕ is an ideal of L, hence a sum of certain simple ideals (Corollary 5.2). Let L' denote the sum of the remaining simple ideals (Theorem 5.2). Then the restriction of ϕ to L' is a faithful representation of L', and we make the preceding construction (using dual bases of L'); the resulting element of End V is again called the Casimir element of ϕ and denoted c_ϕ. Evidently it commutes with $\phi(L) = \phi(L')$, etc.

One last remark: It is often convenient to assume that we are dealing with a faithful representation of L, which amounts to studying the representations of certain (semisimple) ideals of L. If L is simple, only the one dimensional module (on which L acts trivially) or the module 0 will fail to be faithful.

6.3. Weyl's Theorem

Lemma. *Let ϕ: $L \to \mathfrak{gl}(V)$ be a representation of a semisimple Lie algebra L. Then $\phi(L) \subset \mathfrak{sl}(V)$. In particular, L acts trivially on any one dimensional L-module.*

Proof. Use the fact that $L = [LL]$ (5.2) along with the fact that $\mathfrak{sl}(V)$ is the derived algebra of $\mathfrak{gl}(V)$. $\square$

Theorem (Weyl). *Let ϕ: $L \to \mathfrak{gl}(V)$ be a (finite dimensional) representation of a semisimple Lie algebra. Then ϕ is completely reducible.*

Proof. We start with the *special case* in which V has an L-submodule W of codimension one. Since L acts trivially on V/W, by the lemma, we may denote this module F without misleading the reader: $0 \to W \to V \to \mathsf{F} \to 0$ is therefore exact. Using induction on dim W, we can reduce to the case where W is an *irreducible* L-module, as follows. Let W' be a proper nonzero submodule of W. This yields an exact sequence: $0 \to W/W' \to V/W' \to \mathsf{F} \to 0$. By induction, this sequence "splits", i.e., there exists a one dimensional L-submodule of V/W' (say $\tilde{W}/W'$) complementary to W/W'. So we get another exact sequence: $0 \to W' \to \tilde{W} \to \mathsf{F} \to 0$. This is like the original situation, except that $\dim W' < \dim W$, so induction provides a (one dimensional) submodule X complementary to W' in $\tilde{W}$: $\tilde{W} = W' \oplus X$. But $V/W' = W/W' \oplus \tilde{W}/W'$. It follows that $V = W \oplus X$, since the dimensions add up to dim V and since $W \cap X = 0$.

Now we may assume that W is irreducible. (We may also assume without loss of generality that L acts faithfully on V.) Let $c = c_\phi$ be the Casimir element of ϕ (6.2). Since c commutes with $\phi(L)$, *c is actually an L-module endomorphism of V*; in particular, $c(W) \subset W$ and Ker c is an L-submodule