Question 1, Exercise 3.5

Part (a): The time-evolved wave function is given by the action of the time-evolution operator on the initial wave function.

$$\Psi = \hat{U}\left|\psi\right\rangle = \boxed{\frac{1}{\sqrt{8\pi}}\phi_0 \exp(-iE_0 t) + \frac{1}{\sqrt{18\pi}}\phi_2 \exp(-iE_2 t)}$$

Part (c): The probability density is $\rho = \Psi^*\Psi$.

$$\begin{aligned}\rho = \Psi^*\Psi &= \left(\frac{1}{\sqrt{8\pi}}\phi_0 \exp(iE_0 t) + \frac{1}{\sqrt{18\pi}}\phi_2 \exp(iE_2 t)\right)\left(\frac{1}{\sqrt{8\pi}}\phi_0 \exp(-iE_0 t) + \frac{1}{\sqrt{18\pi}}\phi_2 \exp(-iE_2 t)\right)\\ &= \frac{1}{8\pi}\phi_0^2 + \frac{1}{18\pi}\phi_2^2 + \frac{1}{12\pi}\phi_0\phi_2\big(\exp(i(E_0-E_2)t) + \exp(i(E_2-E_0)t)\big)\end{aligned}$$

The exponential $\exp(i\xi) + \exp(-i\xi)$ can be simplified to $2\cos(\xi)$ where $\xi = (E_2 - E_0)t$

$$\rho = \frac{1}{8\pi}\phi_0^2 + \frac{1}{18\pi}\phi_2^2 + \frac{1}{6\pi}\phi_0\phi_2\cos((E_2-E_0)t) = \boxed{\frac{1}{8\pi}\phi_0^2 + \frac{1-2x^2}{18\pi}\phi_0^2 + \frac{1-2x^2}{6\pi}\phi_0^2\cos((E_2-E_0)t)}$$

The probability current is given by $\frac{i}{2}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)$. One way to calculate the derivatives Ψ and Ψ^* is to write ϕ_2 in terms of ϕ_0.

$$\Psi(x,t) = \frac{1}{\sqrt{8\pi}}\phi_0\exp(-iE_0 t) + \frac{1-2x^2}{\sqrt{18\pi}}\phi_0\exp(-iE_2 t) \qquad \Psi^*(x,t) = \frac{1}{\sqrt{8\pi}}\phi_0\exp(iE_0 t) + \frac{1-2x^2}{\sqrt{18\pi}}\phi_0\exp(iE_2 t)$$

The derivative of ϕ_0 is $-x\phi_0$:

$$\frac{\mathrm{d}\phi_0}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\exp\left(-\frac{x^2}{2}\right)\right) = \exp\left(-\frac{x^2}{2}\right)\left(-\frac{2x}{2}\right) = -x\exp\left(-\frac{x^2}{2}\right) = -x\phi_0$$

The derivatives of Ψ and Ψ^* are:

$$\frac{\partial\Psi}{\partial x} = \frac{1}{\sqrt{8\pi}}\frac{\mathrm{d}\phi_0}{\mathrm{d}x}\exp(-iE_0 t) + \frac{1}{\sqrt{18\pi}}\exp(-iE_2 t)\frac{\mathrm{d}}{\mathrm{d}x}\big((1-2x^2)\phi_0\big)$$

$$= \frac{-x}{\sqrt{8\pi}}\phi_0\exp(-iE_0 t) + \frac{1}{\sqrt{18\pi}}\exp(-iE_2 t)\big((1-2x^2)(-x\phi_0) + \phi_0(-4x)\big) = \frac{-x}{\sqrt{8\pi}}\phi_0\exp(-iE_0 t) + \frac{2x^3-5x}{\sqrt{18\pi}}\phi_0\exp(-iE_2 t)$$

$$\frac{\partial\Psi^*}{\partial x} = \left(\frac{\partial\Psi}{\partial x}\right)^* = \frac{-x}{\sqrt{8\pi}}\phi_0\exp(iE_0 t) + \frac{2x^3-5x}{\sqrt{18\pi}}\phi_0\exp(iE_2 t)$$

The products between the wave functions and their derivatives are:

$$\begin{aligned}\Psi\frac{\partial\Psi^*}{\partial x} &= \left(\frac{1}{\sqrt{8\pi}}\phi_0\exp(-iE_0 t) + \frac{1-2x^2}{\sqrt{18\pi}}\phi_0\exp(-iE_2 t)\right)\left(\frac{-x}{\sqrt{8\pi}}\phi_0\exp(iE_0 t) + \frac{2x^3-5x}{\sqrt{18\pi}}\phi_0\exp(iE_2 t)\right)\\ &= \frac{-x}{8\pi}\phi_0^2 + \frac{(1-2x^2)(2x^3-5x)}{18\pi}\phi_0^2 + \frac{2x^3-5x}{12\pi}\phi_0^2\exp(i(E_2-E_0)t) + \frac{2x^3-x}{12\pi}\phi_0^2\exp(i(E_0-E_2)t)\end{aligned}$$

$$\begin{aligned}\Psi^*\frac{\partial\Psi}{\partial x} &= \left(\frac{1}{\sqrt{8\pi}}\phi_0\exp(iE_0 t) + \frac{1-2x^2}{\sqrt{18\pi}}\phi_0\exp(iE_2 t)\right)\left(\frac{-x}{\sqrt{8\pi}}\phi_0\exp(-iE_0 t) + \frac{2x^3-5x}{\sqrt{18\pi}}\phi_0\exp(-iE_2 t)\right)\\ &= \frac{-x}{8\pi}\phi_0^2 + \frac{(1-2x^2)(2x^3-5x)}{18\pi}\phi_0^2 + \frac{2x^3-5x}{12\pi}\phi_0^2\exp(i(E_0-E_2)t) + \frac{2x^3-x}{12\pi}\phi_0^2\exp(i(E_2-E_0)t)\end{aligned}$$

The probability current involves calculating $\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}$.

$$\begin{aligned}&\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\\ &= \frac{2x^3-5x}{12\pi}\phi_0^2\big(\exp(i(E_2-E_0)t) - \exp(i(E_0-E_2)t)\big) + \frac{2x^3-x}{12\pi}\phi_0^2\big(\exp(i(E_0-E_2)t) - \exp(i(E_2-E_0)t)\big)\\ &= \frac{2x^3-5x}{12\pi}\phi_0^2\big(\exp(i(E_2-E_0)t) - \exp(i(E_0-E_2)t)\big) - \frac{2x^3-x}{12\pi}\phi_0^2\big(\exp(i(E_2-E_0)t) - \exp(i(E_0-E_2)t)\big)\\ &= -\frac{x}{3\pi}\phi_0^2\big(\exp(i(E_2-E_0)t) - \exp(i(E_0-E_2)t)\big)\end{aligned}$$

The exponential $\exp(i\xi) - \exp(-i\xi)$ can be simplified to $2i\sin(\xi)$ where $\xi = (E_2 - E_0)t$. The probability current involves multiplying the previous term by $\frac{i}{2}$ (where $\hbar = m = 1$ since this is a dimensionless system).

$$J = \frac{i}{2}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right) = \frac{i}{2}\left(-\frac{x}{3\pi}\phi_0^2 2i\sin((E_2 - E_0)t)\right) = \boxed{\frac{x}{3\pi}\phi_0^2\sin((E_2 - E_0)t)}$$

Part (d): The continuity equation involves $\frac{\partial\rho}{\partial t}$ and $\frac{\partial J}{\partial x}$. The first two terms vanish as they are solely dependent on ϕ_0 which is independent of t. From the previous problem, $E_2 = \frac{5}{2}$ and $E_0 = \frac{1}{2}$ so $E_2 - E_0 = 2$. $\frac{\partial\rho}{\partial t}$ is:

$$\begin{aligned}\frac{\partial\rho}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{8\pi}\phi_0^2 + \frac{1-2x^2}{18\pi}\phi_0^2 + \frac{1-2x^2}{6\pi}\phi_0^2\cos((E_2 - E_0)t)\right) = \frac{\partial}{\partial t}\left(\frac{1-2x^2}{6\pi}\phi_0^2\cos((E_2 - E_0)t)\right)\\ &= \frac{2x^2-1}{6\pi}\phi_0^2\sin((E_2 - E_0)t)(E_2 - E_0) = \frac{2x^2-1}{3\pi}\phi_0^2\sin((E_2 - E_0)t)\end{aligned}$$

The calculation for $\frac{\partial J}{\partial x}$ is:

$$\frac{\partial J}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{3\pi}\phi_0^2\sin((E_2 - E_0)t)\right) = \frac{1}{3\pi}\sin((E_2 - E_0)t)\frac{\mathrm{d}}{\mathrm{d}x}\left(x\phi_0^2\right)$$

The derivative is:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\phi_0^2\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(x\exp\left(-x^2\right)\right) = x\exp\left(-x^2\right)(-2x) + \exp\left(-x^2\right) = \left(-2x^2+1\right)\phi_0^2 = -\left(2x^2-1\right)\phi_0^2$$

Substituting this in for $\frac{\partial J}{\partial x}$:

$$\frac{\partial J}{\partial x} = -\frac{2x^2-1}{3\pi}\phi_0^2\sin((E_2 - E_0)t)$$

The continuity equation is:

$$\frac{\partial\rho}{\partial t} + \frac{\partial J}{\partial x} = \frac{2x^2-1}{3\pi}\phi_0^2\sin((E_2 - E_0)t) - \frac{2x^2-1}{3\pi}\phi_0^2\sin((E_2 - E_0)t) = \boxed{0}$$